Wave turbulence. Les Houches, August 2017.

References (which contain many others)

- Wave turbulence: A story far from over. ACN, Benno Rumpf. World Scientific Nonlinear Series A. Volume 83. Advances in Wave Turbulence pp 1-52. Eds. V. Shrira, SV Nazarenko (2013).
- Rossby and drift wave turbulence & zonal flows. Connaughton, Nazarenko, Quinn arXiv: 1407.1896v1 [physics. flu-dyn] July 7th 2014.
  Large scale drift & Rossby wave turbulence. Harper, Nazarenko. New J. Phys 18 (2016) 085008

- Outline of lecture.
  - What is wave turbulence? Closure ✓ Solutions ✓ Weaknesses ??
  - A solved problem? Closure ✓ Solutions ✓ Weaknesses ??
  - Outline of theory + premises on which closure is based. Connections with the over and misused phrase "random phase approximation."
  - Kinetic equation; its equipartition and Kolmogorov-Zakharov (KZ) solutions. Consequences of nonuniformity in k space of validity of KZ solns. Notions of finite / infinite capacities. Whitecapping, collapses and cycle of intermittency.
  - Detailed derivation of wave turbulence closure for Rossby waves are given in accompanying notes.
  - Open questions and challenges.

Rossby wave equation (Charney, Hasegawa, Mima)

Conservation of the potential vorticity of a column of fluid between two isentropic surfaces of vertical distance $h+\eta$, $\eta \ll h$,

(1) $$\frac{D}{Dt}\left(\frac{f(=2\Omega\cos\theta) - \nabla^2\psi}{h+\eta}\right) = 0$$

where Ω is earth's rotation, θ co-latitude, horizontal velocity $\vec{u} = (u, v) = (\psi_y, -\psi_x)$, $\eta = -f/g\,\psi$ (from hydrostatic balance) $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$, $\frac{df}{dy} = \beta$.

x is east-west, y is north-south,

To quadratic order in nonlinearity,

(2) $$\frac{\partial}{\partial t}(\nabla^2 - \alpha^2)\psi + \beta\psi_x = J(\psi, \nabla^2\psi), \quad J(u,v) = u_x v_y - u_y v_x$$

Let $\psi(\vec{x},t) = \int A(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}$, $A(\vec{k},t) = \frac{1}{(2\pi)^2}\int \psi(\vec{x},t)\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x}$

Because ψ is bounded at $\vec{\infty}$, $A(\vec{k},t)$ is a generalized function of k. $\omega_k = \frac{-\beta k_x}{\alpha^2+k^2} = -\omega_{-k}$

(3) $$A_{kt} + i\omega_k A_k = \int H(\vec{k},\vec{k}_1,\vec{k}_2)\, A(\vec{k}_1) A(\vec{k}_2)\, \delta(\vec{k}_1+\vec{k}_2-\vec{k})\, d\vec{k}_1 d\vec{k}_2,$$

$\vec{k}_1 \times \vec{k}_2 = \vec{k}_1 \times \vec{k} = \vec{k}\times\vec{k}_2$

$H(\vec{k},\vec{k}_1,\vec{k}_2) \triangleq H_{012} = \frac{(\vec{k}_1\times\vec{k}_2)(k_1^2 - k_2^2)}{\alpha^2+k^2}$. Note, because $\vec{k}_1+\vec{k}_2 = \vec{k}$,

Whereas A_k is generalized function, its product averages can be written as the product of Dirac delta functions and "good" functions.

We assume $\psi(\vec{x},t)$ is a spatially random function with zero mean. $\langle\psi\rangle = 0$.

$\langle\psi(\vec{x})\psi(\vec{x}+\vec{r})\rangle = R^{(2)}(\vec{r})$. Since $\langle\psi(\vec{x}-\vec{r})\psi(\vec{x})\rangle = R^{(2)}(-r) = R^{(2)}(\vec{r})$ is real & symmetric.

$$\therefore \langle A(\vec{k})A(\vec{k}')\rangle = \frac{1}{(2\pi)^4}\int \underset{\langle\psi(\vec{x})\psi(\vec{x}')\rangle}{\underset{\parallel}{R^{(2)}(\vec{x}'-\vec{x})}}\, e^{-i\vec{k}\cdot(\vec{x}'-x)} \cdot e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}\, \underset{d\vec{x}\, d(\vec{x}'-\vec{x})}{\underset{\parallel}{d\vec{x}\, d\vec{x}'}} = \frac{\delta(\vec{k}+\vec{k}')}{(2\pi)^2}\int R^{(2)}(\vec{r})\, e^{-i\vec{k}'\cdot\vec{r}}\, d\vec{r}$$

↓ integrate over $\vec{x}$

$$= \delta(\vec{k}+\vec{k}')\, \Phi^{(2)}(\vec{k}') \quad \text{where } \Phi^{(2)}(\vec{k}') = \frac{1}{(2\pi)^2}\int R^{(2)}(\vec{r})\, e^{-i\vec{k}'\cdot\vec{r}}\, d\vec{r}$$ is an ordinary function

Since ψ is real, $R^{(2)}(\vec{r})$ real & sym. $\Rightarrow$ $\Phi^{(2)}(\vec{k}) = \Phi^{(2)}(-k) = \Phi^{(2)*}(k)$.

Energy density $\left(\langle \frac{1}{2}\rho(\psi_x^2 + \psi_y^2 + \alpha^2\psi^2)\rangle\right)$ in $\vec{k}$ space $e(\vec{k}) = \int (\alpha^2+k^2)\,\Phi^{(2)}(k)\, d\vec{k}$.

↓ potential $\rho\int_0^{h+\eta} gz\, dz \cdot dx\, dy$

Likewise

$$\langle A(k)\,A(k')\,A(k'')\rangle = \delta(\vec{k}+\vec{k}'+\vec{k}'')\,\varphi^{(3)}(\vec{k},\vec{k}',\vec{k}'')$$

$$\langle A(k)\,A(k')\,A(k'')\,A(k''')\rangle = \delta(\vec{k}+\vec{k}'+\vec{k}''+\vec{k}''')\,\varphi^{(4)}(\vec{k},\vec{k}',\vec{k}'',\vec{k}''') + \delta(k+k')\,\delta(k''+k''')\,\varphi^{(2)}(k')\,\varphi^{(2)}(k''') + \delta(k+k'')\,\delta(k'+k''')\,\varphi^{(2)}(k'')\,\varphi^{(2)}(k''') + \delta(k+k''')\,\delta(k'+k'')\,\varphi^{(2)}(k'')\,\varphi^{(2)}(k''').$$

We use cumulants rather than moments because their physical transforms $R^{(2)}(\vec{r})$, $R^{(3)}(\vec{r},\vec{r}')$, $R^{(4)}(\vec{r},\vec{r}',\vec{r}'')$ have the property that $R^{(n)}\to 0$ as the arguments $\vec{r},\vec{r}',\dots\to\infty$

Therefore, at least initially, $\varphi^{(n)}$ are "good" functions; i.e. have ordinary FT's

— at least initially!

Wave turbulence cumulant hierarchy

Multiply ③ by $A_{k'}$, mult. ③ with k replaced by k' by A_k, and add to get

$$④\ \delta(\vec{k}+\vec{k}')\cdot\quad \frac{d\varphi^{(2)}(k')}{dt} + \underbrace{i(\omega_k+\omega_{k'})}_{=0}\,\varphi^{(2)}(k') = \underbrace{\overset{00'}{P}}_{\text{permutation over } 0,0'} \int \underbrace{H_{012}}_{H(\vec{k},\vec{k}_1,\vec{k}_2)}\,\varphi^{(3)}(k',k_1,k_2)\,\underbrace{\delta_{12,0}}_{\delta(\vec{k}_1+\vec{k}_2-\vec{k})}\,\underbrace{dk_{12}}_{dk_1\,dk_2}$$

Similarly, multiply ③ by $A_{k'}A_{k''}$, $\overset{00'0''}{P}$ (+add) to obtain

$$⑤\ \delta(\vec{k}+\vec{k}'+\vec{k}''):\quad \frac{d\,\varphi^{(3)}(\vec{k},\vec{k}',\vec{k}'')}{dt} + i(\omega_k+\omega_{k'}+\omega_{k''})\,\varphi^{(3)}(\vec{k},\vec{k}',\vec{k}'') = \overset{00'0''}{P}\left\{\int H_{012}\,\varphi^{(4)}(k',k'',k_1,k_2)\,\delta_{0,12}\,dk_{12} + 2H_{0\text{-}0'\text{-}0''}\,\varphi^{(2)}(k')\,\varphi^{(2)}(k'')\right\}$$

Continue in this way to find eqns. for $\varphi^{(n)}$, $n = 2, 3, 4, 5, \dots$.

The resulting hierarchy is not closed. This is the main challenge facing theoretical approaches to turbulence. How to effect a consistent closure!!

In wave turbulence, we take advantage of both ⓐ weak nonlinearity and ⓑ The Riemann-Lebesgue lemma, which uses the fact that the waves are dispersive, to bring about a natural asymptotic closure.

Asymptotic closure of the wave turbulence hierarchy.

Because the nonlinearity is weak, the RHS's of (4) and (5) are multiplied by "ε", $0<\varepsilon\ll 1$. Therefore it is natural to solve the hierarchy (4), (5), ... iteratively as

(6) $\varphi^{(2)} = \varphi_0^{(2)} + \varphi_1^{(2)} + \varphi_2^{(2)}, \dots \quad \varphi^{(3)} = \varphi_0^{(3)} + \varphi_1^{(3)} + \dots$ etc.

and then look at the long time behaviors of the iterates $\varphi^{(n)}_{1,2,\dots}$ etc.

The closure arises because the secular terms in $\varphi_2^{(2)}, \varphi_2^{(3)}, \dots$ depend respectively only on $\varphi_0^{(2)}$ and products of $\varphi_0^{(n)} \varphi_0^{(2)}$ for $n\geq 3$.

We remove the secular terms from $\varphi_2^{(n)}, n\geq 2$, by choosing the slow time behaviors of $q_0^{(2)}(\vec{k}) = \varphi_0^{(2)}(k)$, $q_0^{(n)}(\vec{k},\vec{k}',\dots) = \varphi_0^{(n)}(\vec{k},\vec{k}',\dots)e^{i(\omega_k t + \omega_{k'} t+\dots)}$

(7) $\dfrac{dq_0^{(n)}}{dt} = F_1^{(n)} + F_2^{(n)} + \dots$, $F_j^{(n)}$ slowly varying in t,

where $F_1^{(n)}$ is chosen to remove secular growths in $\varphi_1^{(n)}$ (there are none) and $F_2^{(n)}$ is chosen to remove secular growths in $\varphi_2^{(n)}$. The resulting equations for $q_0^{(n)}$ are closed.

These calculations are usually most easily carried out by iterating directly the equation for $a_k = A_k e^{i\omega_k t}$ and then forming the iterates $q_1^{(n)}, q_2^{(n)}$ from the averages $\langle a_{0k} a_{1k'} \rangle$, $\langle a_{0k} a_{2k'} + a_{1k} a_{2k'} \rangle$, ... and this we will do on pgs. . But in order to bypass all the notation this requires, we will first work directly with (4) and (5) and show that the secular terms in $\varphi_2^{(2)}$ only involve $q_0^{(2)}$ and not $q_0^{(3)}$ nor $q_0^{(4)}$. For Rossby waves, secular terms have the algebraic behavior "$\varepsilon^2 t$".

R.4.

At leading order (i.e. to order one ε^0)

(4) $$\frac{dq_0^{(2)}}{dt} = 0$$

(5) $$\frac{dq_0^{(n)}}{dt} = 0, \quad \varphi_0^{(n)}(\vec{k}, \vec{k}', \ldots) = q_0^{(n)}(\vec{k}, \vec{k}', \ldots) e^{-i(\omega_k + \omega_{k'} + \ldots)t}.$$

At first order (i.e. to order ε^1)

(4) $$\frac{dq_1^{(2)}}{dt} = -F_1^{(2)} + \overset{00'}{P} \int H_{012}\, q_0^{(3)}(k', k_1, k_2)\, e^{i(\omega-\omega_1-\omega_2)t} \delta_{12,0}\, d12$$ $(\omega_k + \omega_{k'} = \omega_k + \omega_{-k} = 0)$.

$F_1^{(2)}$ from $\frac{dq_0^{(2)}}{dt}$

where $\overset{00'}{P}$ is the permutation over $\vec{k}, \vec{k}'$.

Integrating the second term we find $\overset{00'}{P} \int H_{012}\, q_0^{(3)}(\vec{k}', \vec{k}_1, \vec{k}_2)\, \Delta(\omega-\omega_1-\omega_2)\, \delta_{12,0}\, d12$

where $\Delta(x) = \int_0^t e^{ixt}\, dt = \frac{e^{ixt}-1}{ix}$. For "smooth" ($\in L_2$) $f(x)$, $\lim_{t\to\infty} \int f(x)\Delta(x)\, dx = \pi\, \mathrm{sgn}\, t\, f(0) + iP\int \frac{f(x)}{x}$ so $\Delta(x) \sim \tilde{\Delta}(x) = \pi\, \mathrm{sgn}\, t\, \delta(x) + iP\left(\frac{1}{x}\right)$.

P → Cauchy principal value

Therefore this term is bounded so we may choose $F_1^{(2)} = 0$.

(5) $$\frac{dq_1^{(3)}}{dt} = -F_1^{(3)} + \overset{00'0''}{P} \Big\{ \int H_{012}\, q_0^{(4)}(k', k'', k_1, k_2)\, e^{i(\omega-\omega_1-\omega_2)t} \delta_{12,0}\, d12 + 2H_{0\text{-}0'\text{-}0''}\, q_0^{(2)}(k')\, q_0^{(2)}(k'')\, e^{i(\omega+\omega'+\omega'')t} \Big\}$$

(8) $$q_1^{(3)} = -F_1^{(3)} t + \overset{00'0''}{P} \Big\{ H_{012}\, q_0^{(4)}(k', k'', k_1, k_2)\, \Delta(\omega-\omega_1-\omega_2)\, \delta_{12,0}\, d12 + 2H_{0\text{-}0'\text{-}0''}\, q_0^{(2)}(k')\, q_0^{(2)}(k'')\, \Delta(\omega+\omega'+\omega'') \Big\}$$

The long time behavior of $\overset{00'0''}{P}\{\ \}$ is bounded so we choose $F_1^{(3)} = 0$.

However, the long time behavior of $q_1^{(3)}$ is also non-smooth because of the last terms.

We call them the asymptotic sumisvors. It is these terms which lead to nontrivial closure. Because the asymptotic sumisvos for $q_1^{(3)}$ at $O(\varepsilon)$ contain only products of $q_0^{(2)}$, for $q_2^{(4)}$ at $O(\varepsilon^2)$ contain only $q_0^{(2)}$ products, a natural closure is achieved without the need for restrictive statistical assumptions.

More on this in my lecture and later in this set of notes.

R.5.

At second order (i.e. order ε^2)

(4)' $$\frac{dq_2^{(2)}}{dt} = -F_2^{(2)} + \overset{00'}{P}\int H_{012}\, q_1^{(3)}(k',k_1,k_2)\, e^{-i(\omega'-\omega_1-\omega_2)t}\,\delta_{12,0}\,d_{12}$$

The part of $q_1^{(3)}$ which involves $q_0^{(4)}$ as given by (8) will, when integrated, give bounded solutions as $t\to\infty$. The part which involves the asymptotic survivors will all lead to terms growing proportional to t. We use $F_2^{(2)}$ to remove these terms. Then $\frac{dq_0^{(2)}}{dt} = F_2^{(2)} = F_2^{(2)}\{q_0^{(2)}\}$ will be a closed equation for $q_0^{(2)}$. It is called the kinetic equation.

From (8) (including only the asymptotic survivor terms with products of $q_0^{(2)}$), and (4)'' above, we obtain (on applying $\overset{00'}{P}$ and then (8))

$$\int H_{012}\, q_1^{(3)}(\underset{-k}{k'},k_1,k_2)\, e^{-i(\omega-\omega_1-\omega_2)t}\,\delta_{12,0}\,d_{12} + \int H_{-0-1-2}\, q_1^{(3)}(k,-k_1,-k_2)\, e^{i(\omega-\omega_1-\omega_2)t}\,\delta_{12,0}\,d_{12}$$

in this integral ↑ we have changed $\vec k_1, \vec k_2 \to -\vec k_1, -\vec k_2$

$$= 2\int H_{012}\Big(H_{-0-1-2}\, q_0^{(2)}(k_1)q_0^{(2)}(k_2) + H_{1-20}\, q_0^{(2)}(k_2)q_0^{(2)}(k) + H_{20-1}\, q_0^{(2)}(k)q_0^{(2)}(k_1)\Big)\Delta(\omega-\omega_1-\omega_2)\,\delta_{12,0}\,d_{12}$$

$$+ 2\int H_{-0-1-2}\Big(H_{012}\, q_0^{(2)}(k_1)q_0^{(2)}(k_2) + H_{-12-0}\, q_0^{(2)}(k_2)q_0^{(2)}(k) + H_{-2-01}\, q_0^{(2)}(k)q_0^{(2)}(k_1)\Big)\Delta(-\omega+\omega_1+\omega_2)\,\delta_{12,0}\,d_{12}$$

where we have used $q_0^{(2)}(-k) = q_0^{(2)}(k)$ and $\Delta(x)e^{-ixt} = \Delta(-x)$. We will also use the property that $H_{-0-1-2} = H_{012}$. Because $\Delta(x) \sim \tilde\Delta(x) = \pi\delta(x) + iP(\frac{1}{x})$, the principal value terms in the two integrals cancel because the arguments of $\Delta(\omega-\omega_1-\omega_2)$ and $\Delta(-\omega+\omega_1+\omega_2)$ are of opposite signs. We choose $F_2^{(2)}$ to remove secular terms and then find:

$\boxed{\text{In ref. 2., } e(k) = |\omega(k)|\, \underline{\underline{n(k)}}:}$

(9). $$\frac{dq_0^{(2)}}{dt} = 4\pi\,\mathrm{sgn}t\int H_{012}\, q_0^{(2)}(k)\,q_0^{(2)}(k_1)\,q_0^{(2)}(k_2)\left\{\frac{H_{012}}{q_0^{(2)}(k)} + \frac{H_{1-20}}{q_0^{(2)}(k_1)} + \frac{H_{20-1}}{q_0^{(2)}(k_2)}\right\}\delta(\omega-\omega_1-\omega_2)\,\delta(\vec k_1+\vec k_2-\vec k)\,d\vec k_1\,d\vec k_2$$

Recall the energy $e(k) = (\alpha^2+k^2)\,q_0^{(2)}(k)$. Then (recall $k_1\times k_2 = k\times k_2 = k_1\times k$)

(10) $$\frac{de(k)}{dt} = 4\pi\,\mathrm{sgn}t\int \frac{(k_1\times k_2)^2(k_1^2-k_2^2)}{(\alpha^2+k_1^2)(\alpha^2+k_2^2)}\, e(k)\,e(k_1)\,e(k_2)\left\{\frac{k_1^2-k_2^2}{e(k)} + \frac{k_2^2-k^2}{e(k_1)} + \frac{k^2-k_1^2}{e(k_2)}\right\}\delta(\omega-\omega_1-\omega_2)\,\delta(\vec k-\vec k_1-\vec k_2)\,d\vec k_1\,d\vec k_2$$

Note the equipartition solutions $e(k) = \text{const.}$, $e(k)(\alpha^2+k^2) = \text{const.}$, energy & enstrophy.

✓S.1.

In carrying out calculations, it may be preferable to compute iterates of cumulants from iterates of $a_k = A_k e^{i\omega_k t}$. From 3,

$$a_{kt} = \int H_{012}\, a_1 a_2\, e^{-iW_{12,0}t}\, \delta_{12,0}\, dk_{12}, \quad W_{0,12} = \omega - \omega_1 - \omega_2, \quad dk_{12} = d\vec{k}_1 d\vec{k}_2$$

<u>Define</u> $\Delta(x) = \int_0^t e^{ixt} dx = \dfrac{e^{ixt}-1}{ix} \underset{t\to\infty}{\sim} \tilde{\Delta}(x) = \pi\, \mathrm{sgn}\, t\, \delta(x) + iP\left(\frac{1}{x}\right)$

$$E(x,y) = \int_0^t \Delta(x-y)\, e^{iyt}\, dt. \quad E(x,0) \sim \tilde{\Delta}(x)\left(t - i\frac{\partial}{\partial x}\right).$$

$$\Delta(x)\Delta(-x) \sim 2\pi t\, \mathrm{sgn}\, t\, \delta(x) + 2P\left(\tfrac{1}{x}\right)\frac{\partial}{\partial x}.$$

i.e. for "smooth" $f(x)$ ($\in L_2$ & diff.) $\lim_{t\to\infty} \int f(x)\Delta(x)\Delta(-x)\, dx = 2\pi t\, \mathrm{sgn}\, t\, f(0) + 2P\int \frac{\frac{\partial}{\partial x}f(x)}{x} dx$ & find ($\overset{12}{P}$ permutation over 1,2)

Iterate: $a = a_0 + a_1 + a_2 + \cdots$

$a_{0t} = 0 \Rightarrow a_0$ slowly varying in time

$$a_{1t} = \int H_{012}\, a_{01} a_{02}\, e^{iW_{0,12}t}\, \delta_{12,0}\, dk_{12} \Rightarrow a_1 = \int H_{012}\, a_{01} a_{02}\, \Delta_{0,12}\, \delta_{12,0}\, dk_{12}$$

$$a_{2t} = \int \overset{12}{P} H_{012}\, a_{11} a_{02}\, e^{iW_{0,12}}\, \delta_{12,0}\, dk_{12} \Rightarrow a_2 = \int \overset{12}{P} H_{012} H_{134}\, a_{02} a_{03} a_{04}\, \delta_{34,1}\, \delta_{12,0}\, E(0,234;\, 0,12)\, dk_{234}$$

Iterates of cumulants $\varphi^{(n)}(k,k',\ldots) = q^{(n)}(k,k',\ldots)\, e^{-i(\omega+\omega'+\cdots)t}$; $\vec{k}+\vec{k}'+\cdots = 0$

$$\delta(k+k')\, q^{(2)}(k,k') = \langle a_k a_{k'}\rangle = \langle a_{0k} a_{0k'}\rangle + \overset{00'}{P}\langle a_{0k} a_{1k'}\rangle + \left(\overset{00'}{P}\langle a_{0k} a_{2k'}\rangle + \langle a_{1k} a_{1k'}\rangle\right) + \cdots$$

$$\delta(k+k'+k'')\, q^{(3)}(k,k',k'') = \langle a_k a_{k'} a_{k''}\rangle = \langle a_{0k} a_{0k'} a_{0k''}\rangle + \overset{0'0'0''}{P}\langle a_{1k} a_{0k'} a_{0k''}\rangle + \overset{0'0'0''}{P}\langle a_{2k} a_{0k'} a_{0k''} + a_{0k} a_{1k'} a_{1k''}\rangle$$

$$\delta(k+k')\, q_1^{(2)}(k,k') = \delta(k+k')\,\overset{00'}{P}\int H_{012}\, q_0^{(3)}(k',k_1,k_2)\,\Delta_{0,12}\,\delta_{12,0}\, dk_{12}$$ bounded as $t\to\infty$ as long as $q_0^{(3)}$ is smooth.

- 2nd order cumulants $\delta(k+k')\, q_2^{(2)}(k,k')$ [$q_2^{(2)}(k)$] $= \overset{00'}{P}\int \overset{12}{P} H_{012} H_{134}\, \langle a_{0k'} a_{02} a_{03} a_{04}\rangle\, \delta_{34,1}\, \delta_{12,0}\, E(0,234;\, 0,12)\, dk_{234}$

$\langle a_{0k'} a_{02} a_{03} a_{04}\rangle$: $\delta(k'+k_2+k_3+k_4)\, q_0^{(4)}(k',k_2,k_3,k_4) + \delta_{0'2}\delta_{34}\, q_0^{(2)}(k')\, q_0^{(2)}(k_2)$ zero ∵ $\delta_{34}\delta_{34,1} \Rightarrow k_1 = 0 \Rightarrow H_{134} = 0$.

$+ 2\,\delta_{0'3}\delta_{24}\, q_0^{(2)}(k)\, q_0^{(2)}(k_2) \to$ Int. over k_3, k_4 leads to $E(0;\, 0,12) \sim t\tilde{\Delta}(x) + O(1)$.

$$+ \int H_{012} H_{0'34}\, \langle a_{01} a_{02} a_{03} a_{04}\rangle\, \Delta_{0,12}\, \Delta_{0',34}\, \delta_{12,0}\, \delta_{34,0'}\, dk_{1234}$$

$\langle a_{01} a_{02} a_{03} a_{04}\rangle$: $\delta(k_1+k_2+k_3+k_4)\, q_0^{(4)}(k_1,k_2,k_3,k_4) + \delta_{12}\delta_{34}\, q_0^{(2)}(k_2)\, q_0^{(2)}(k_4) \to 0$

$+ 2\, q_0^{(2)}(k_3)\, q_0^{(2)}(k_4)\, \delta_{13}\, \delta_{24} \to$ leads to secular term ∵ $\Delta_{0,12}\, \Delta_{12,0} \sim 2\pi t\, \mathrm{sgn}\, \delta(x)$

> Note: Principal value secular terms in $\langle a_0 a_2\rangle$, vanish on application of $\overset{00'}{P}$ in 2nd $\int$, $k_1, k_2 \to -\vec{k}_1, -\vec{k}_2$

$$q_2^{(2)}\Big|_{\text{secular}} = 4\pi t\, \mathrm{sgn}\, t \int \Big(H_{012}^2\, q_0^{(2)}(k_1)\, q_0^{(2)}(k_2) + H_{012}\, q_0^{(2)}(k)\big(H_{10-2}\, q_0^{(2)}(k_2) + H_{20-1}\, q_0^{(2)}(k_1)\big)\Big)\, \delta(\omega-\omega_1-\omega_2)\, \delta(\vec{k}-\vec{k}_1-\vec{k}_2)\, d\vec{k}_{12}$$

H_{012}^2 ∵ $H_{-0-1-2} = H_{012}$: +ive feed

negative (losses from k via resonances)

✓ S.2.

- $\delta(k+k'+k'')\left(q_1^{(3)}(k,k',k'')+q_2^{(3)}(k,k',k'')\right) = \overset{00'0''}{P}\langle a_{0k}a_{0k'}a_{0k''}\rangle + \overset{00'0''}{P}\langle a_{2k}a_{0k'}a_{0k''}+a_{0k}a_{1k'}a_{1k''}\rangle$

The first term leads to <u>bounded</u> but <u>nonsmooth</u> contributions discussed in . The second term will give us secular (t growth) behavior which we now calculate.

$$\overset{00'0''}{P}\left\{\int \overset{12}{P} H_{012}H_{134}\langle a_{0k'}a_{0k''}a_{02}a_{03}a_{04}\rangle\, E(0,234;0,12)\,\delta_{012}\,\delta_{34,1}\,d1234\right.$$
$$\left.+\int H_{0'12}H_{0''34}\langle a_{0k}a_{01}a_{02}a_{03}a_{04}\rangle\,\Delta_{0',12}\,\Delta_{0'',34}\,\delta_{12,0'}\,\delta_{34,0''}\,d1234\right\}$$

The only secular term arises from $\overset{34}{P}\langle a_{0k'}a_{0k''}a_{03}\rangle\langle a_{02}a_{04}\rangle$ in the 1st integral as it terms in decomposition of $\langle a_{0k'}a_{0k''}a_{02}a_{03}a_{04}\rangle$ in the only way we get $E(0;0,12)$.

$$q_2^{(3)}\Big|_{secular} \overset{\text{assume } t>0}{=} q_0^{(3)}(k,k',k'')\,2t\,\overset{00'0''}{P}\left(\int H_{012}\left(H_{10-2}\,q_0^{(2)}(k_2)+H_{20-1}\,q_0^{(2)}(k_1)\right)\left(\pi\delta(\omega-\omega_1-\omega_2)+iP\frac{1}{\omega-\omega_1-\omega_2}\right)\delta_{012}\,d12\right).$$

Secular terms are removed by choices of $F_2^{(2)}$ and $F_2^{(3)}$ giving

$$\frac{dq_0^{(2)}}{dt} \overset{t>0}{=} 4\pi\int H_{012}\,q_0^{(2)}(k)\,q_0^{(2)}(k_1)\,q_0^{(2)}(k_2)\left(\frac{H_{-0-1-2}}{q_0^{(2)}(k)}+\frac{H_{10-2}}{q_0^{(2)}(k_1)}+\frac{H_{20-1}}{q_0^{(2)}(k_2)}\right)\delta(\omega-\omega_1-\omega_2)\,\delta(\vec{k}-\vec{k}_1-\vec{k}_2)\,d\vec{k}_{12}$$

$$\frac{1}{q_0^{(3)}}\frac{dq_0^{(3)}}{dt} = 2\overset{00'0''}{P}\int H_{012}\left(H_{10-2}\,q_0^{(2)}(k_2)+H_{20-1}\,q_0^{(2)}(k_1)\right)\left(\pi\delta(\omega-\omega_1-\omega_2)+iP\frac{1}{\omega-\omega_1-\omega_2}\right)\delta_{012}\,d12.$$

It is equivalent to (recall $\Phi_0^{(3)}=q_0^{(3)}e^{-i(\omega+\omega'+\omega'')t}$

$$\omega_k \to \omega_k - 2\int H_{012}\left(H_{10-2}\,q_0^{(2)}(k_2)+H_{20-1}\,q_0^{(2)}(k_1)\right)\left(P\frac{1}{\omega-\omega_1-\omega_2}-i\pi\delta(\omega-\omega_1-\omega_2)\right)\delta_{012}\,d12$$

Can show $\mathrm{Im}\,\omega_k<0$ s.t., due to resonances, higher order cumulants to leading order decay.

One can show the same result holds for all higher order cumulants

$$\frac{1}{q_0^{(n)}}\frac{dq_0^{(n)}}{dt} = 2\overset{00'0''\cdots}{P}\int H_{012}\left(H_{10-2}\,q_0^{(2)}(k_2)+H_{20-1}\,q_0^{(2)}(k_1)\right)\left(\pi\delta(\omega-\omega_1-\omega_2)+iP\frac{1}{\omega-\omega_1-\omega_2}\right)\delta_{012}\,d12.$$

(*) Whereas leading order terms decay, $q^{(n)}$ has asymptotic survivors at $O(\varepsilon^{n-2})$