Instabilities of Internal Wave Beams to study *Abyssal mixing in the laboratory?*

Thierry Dauxois

CNRS

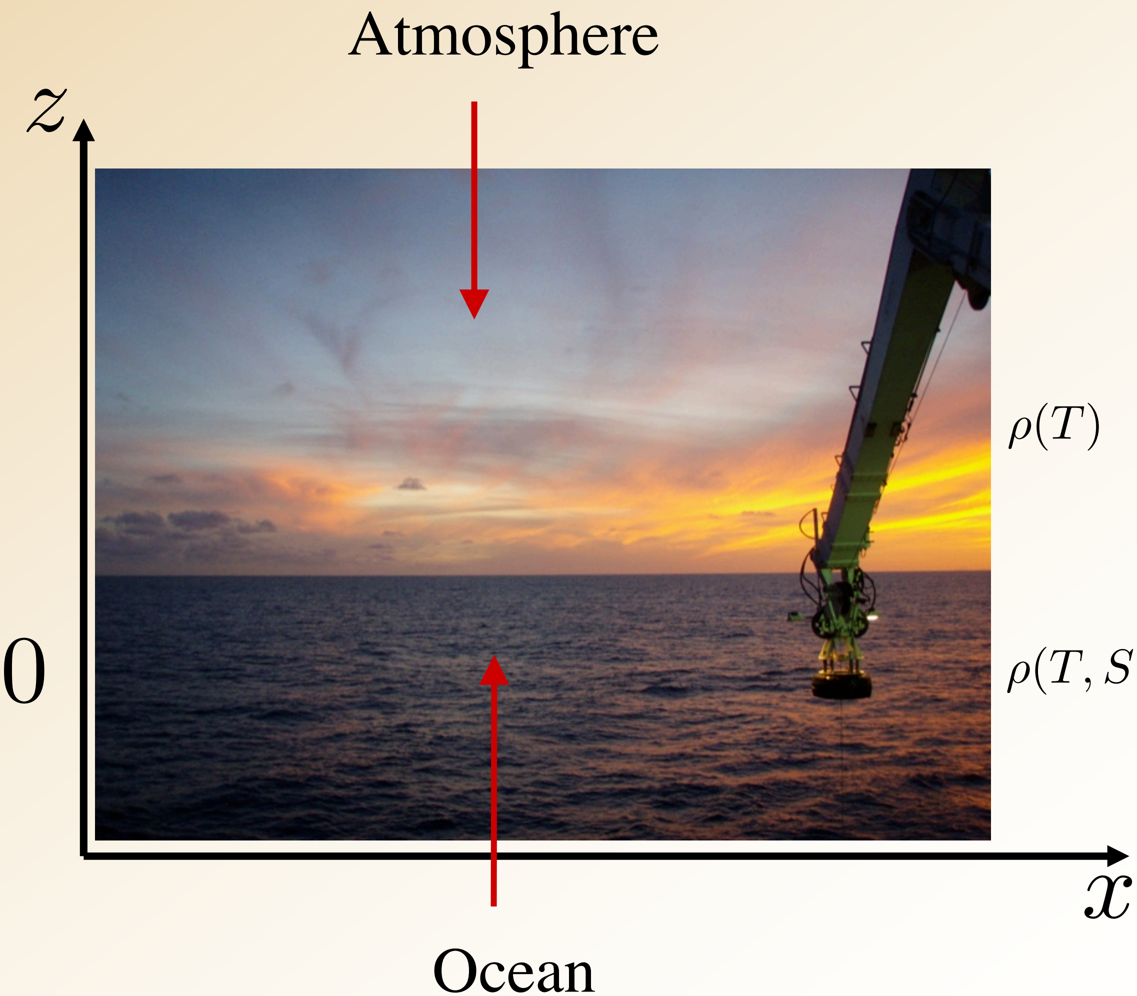
ENS de Lyon

Lab PHYS ENS de LYON

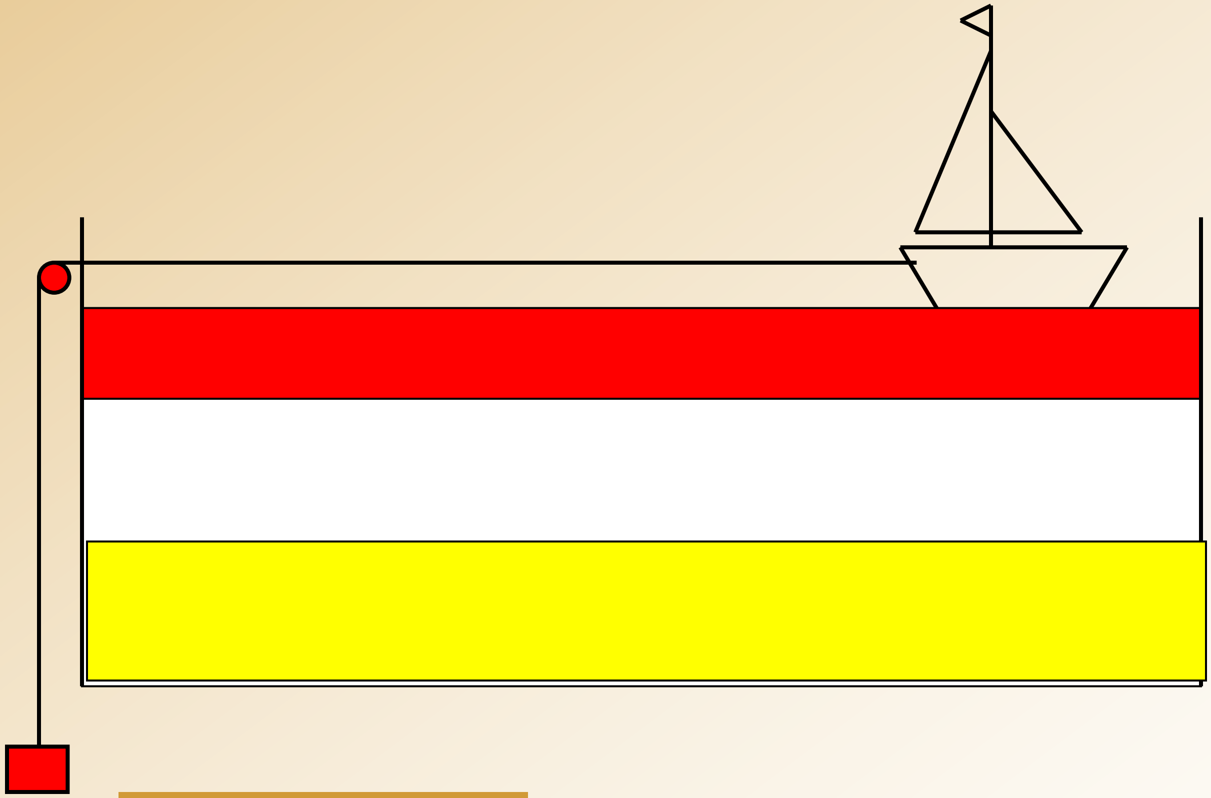
Stratified Fluids

Density $\rho(z)$

Stability $\dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z} < 0$



Internal Waves at a Density Interface

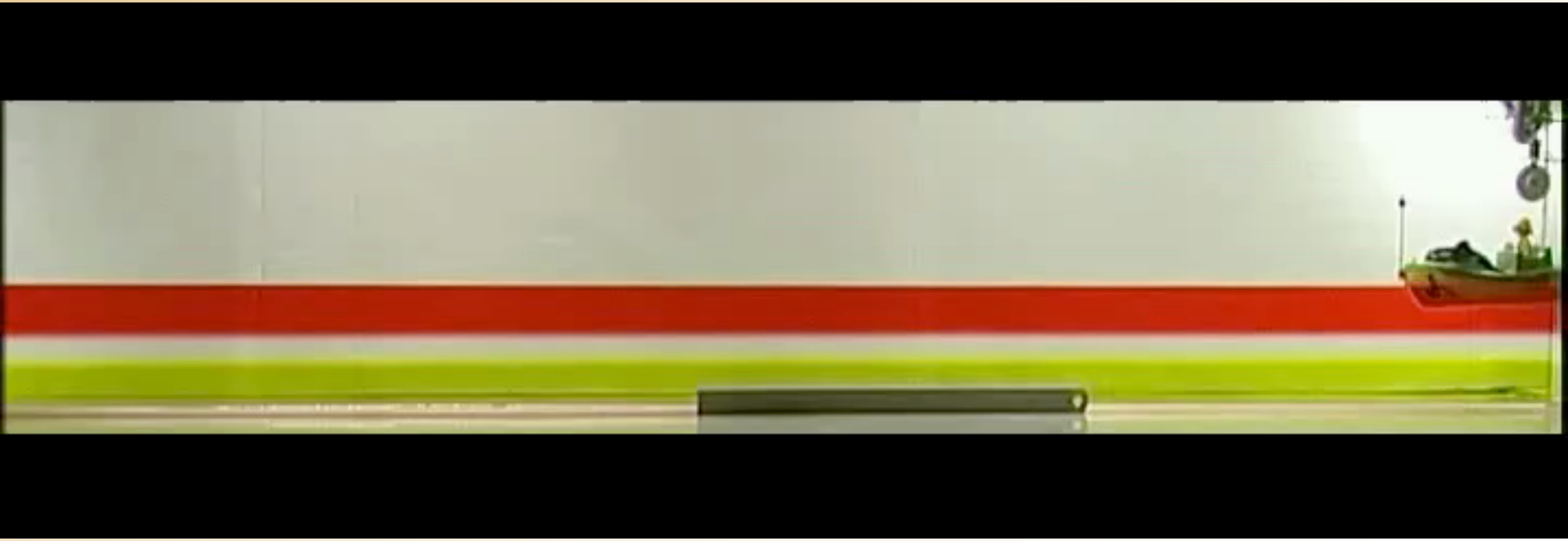


Parameters:

- Tension
- Weight of the boat
- Depths of layers
- Difference of densities

Ekman 1904

Internal Waves at a Density Interface

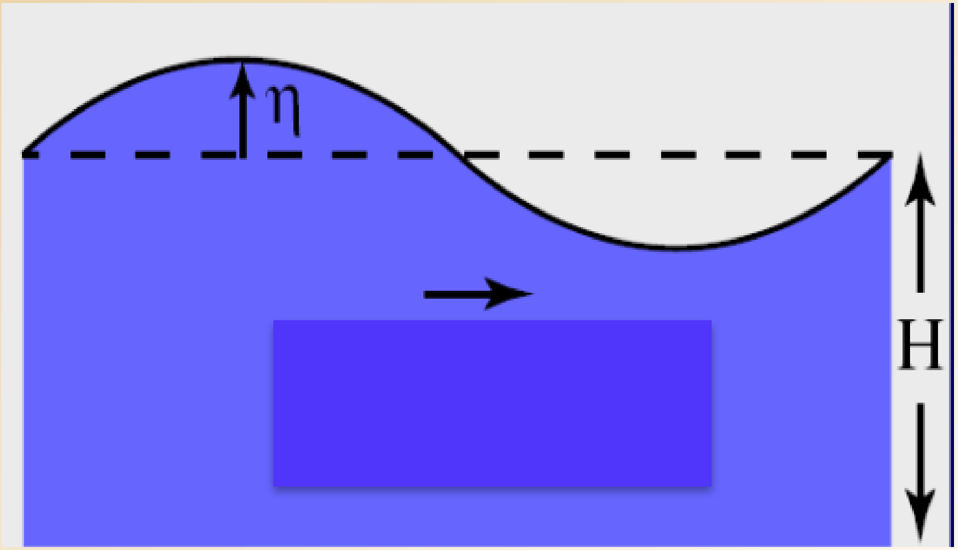


Mercier Vasseur

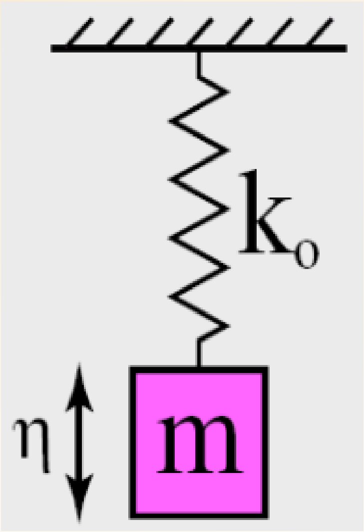
Mercier, Vasseur, Dauxois,
Resurrecting Dead-Water Phenomenon,
Nonlinear Processes in Geophysics **18**, 193-208 (2011).

Internal Waves at a Density Interface

Surface Gravity Waves

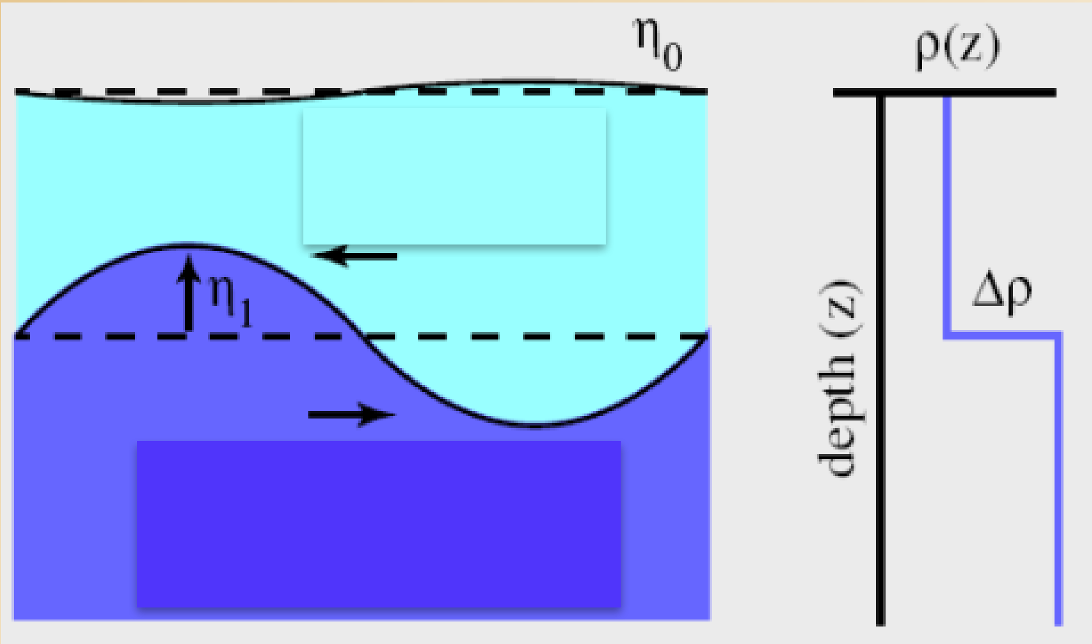


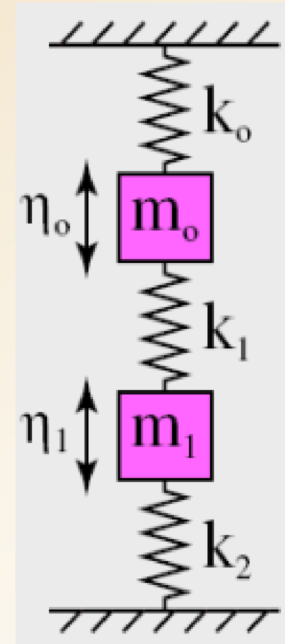
Mass/ Spring



Internal Waves at a Density Interface

Two Layers

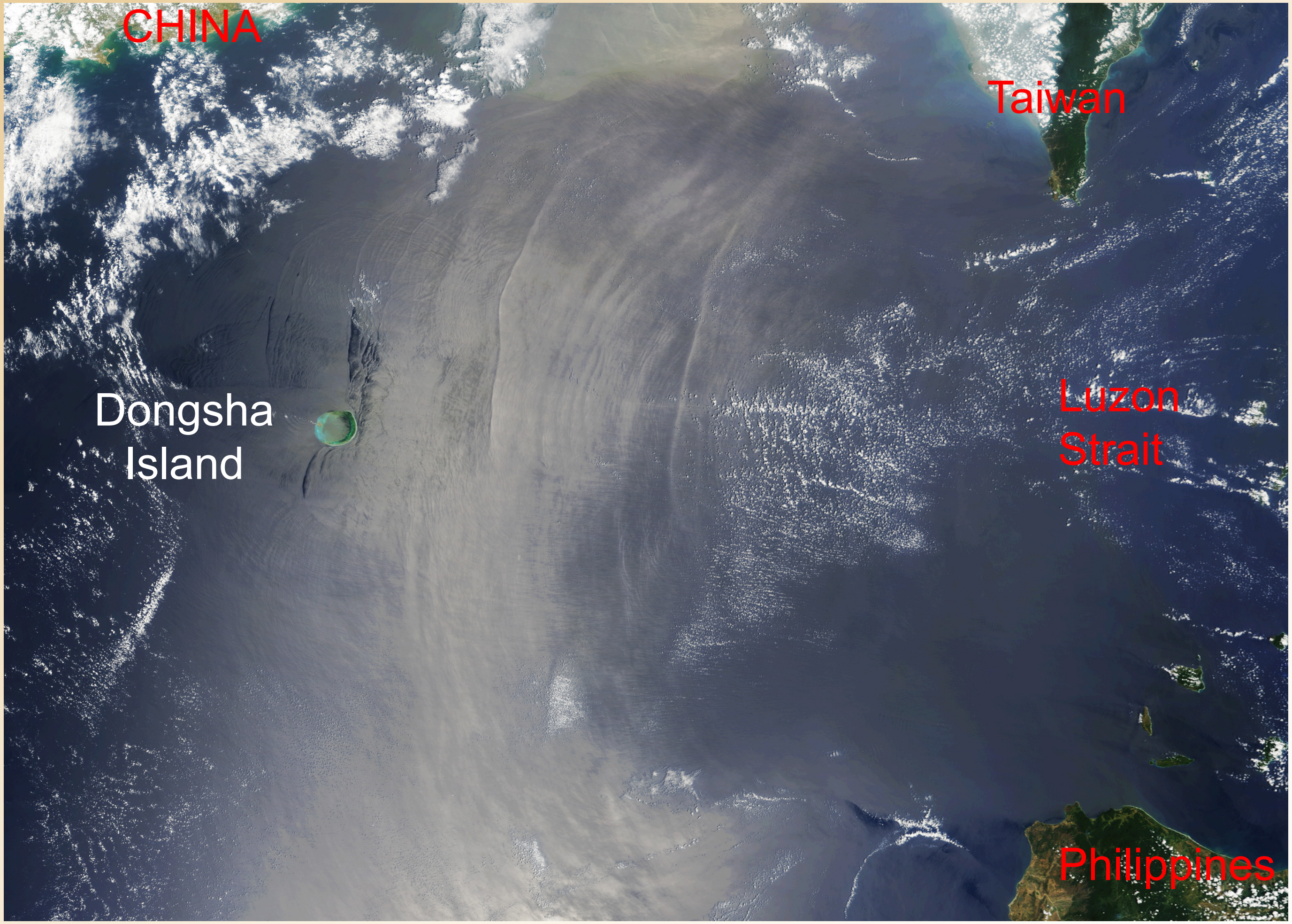




- in the ocean $\Delta\rho/\rho \sim 1/1000$
- if similar velocities in both layers → $\eta_1 \sim 1000\eta_0$
- 3 cm of internal displacement → 30 μm surface expression
- 100 m →

Large amplitude internal waves

South China Sea

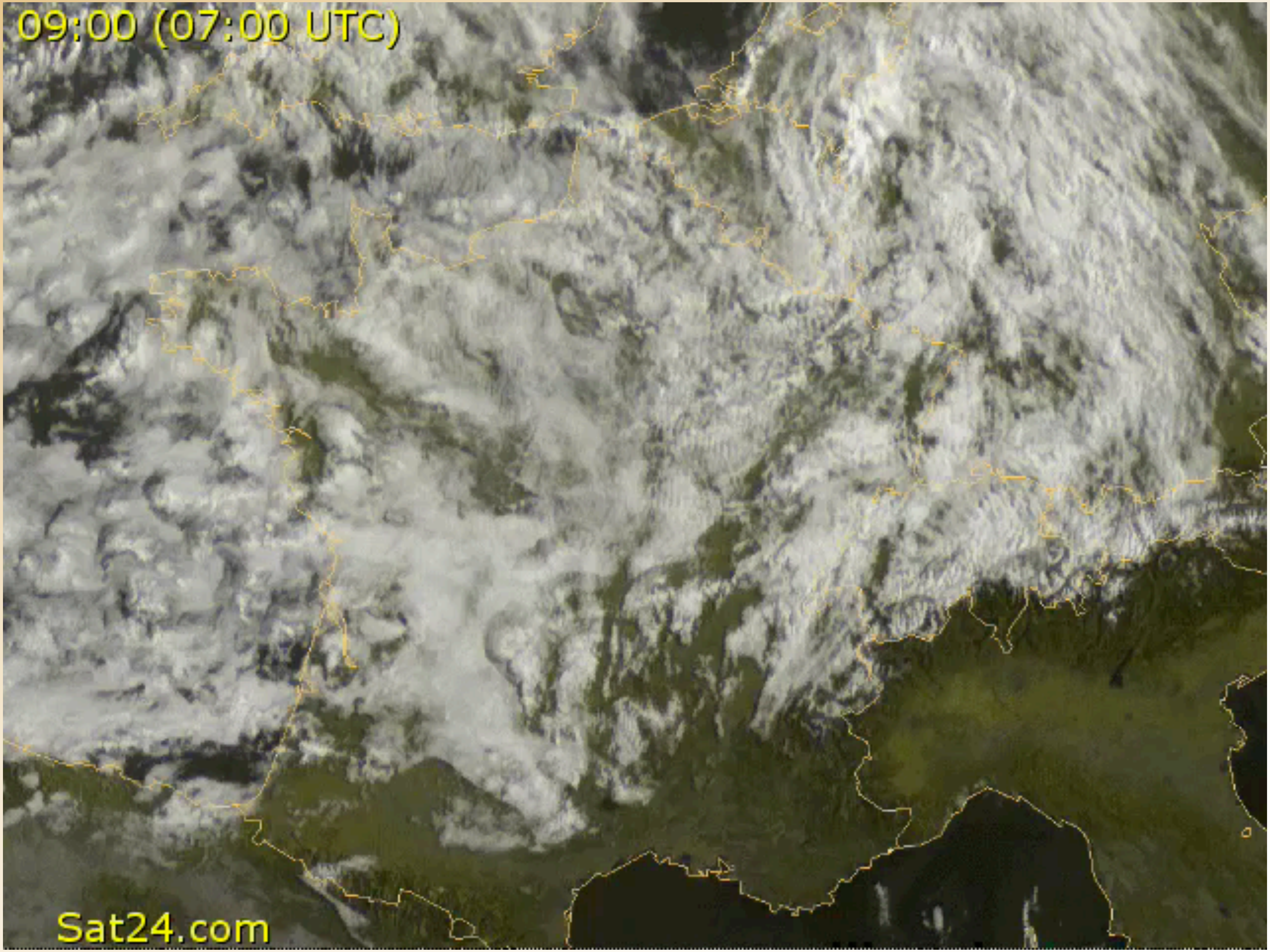


MODIS image, courtesy of NASA

Mercier, Mathur, Gostiaux, Gerkema, Magalhaess, Da Silva, Dauxois, *Journal of Fluid Mechanics* 704, 37 (2012) -
Soliton generation by internal tidal beams impinging on a pycnocline : laboratory experiments.

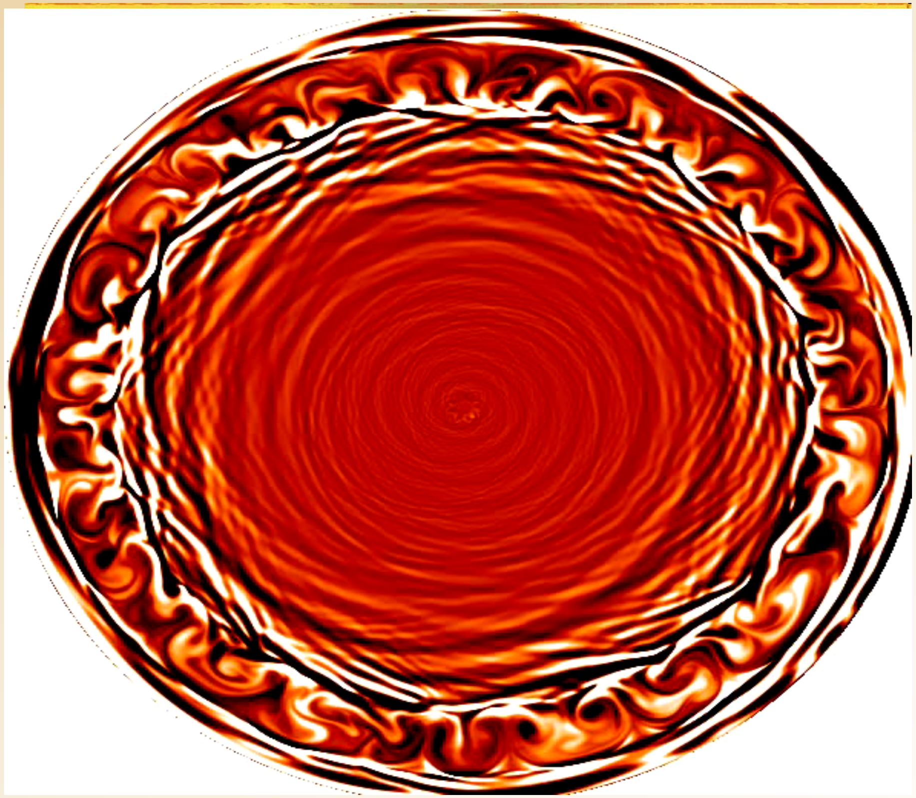
Mercier, Gostiaux, Helfrich, Sommeria, Viboud, Didelle, Ghaemsaidi, Dauxois, Peacock, *Geophysical Research Letters* (2013)
Large-scale, realistic laboratory modeling of M2 internal tide generation at the Luzon Strait.

Atmosphere



Atmosphere

Stars

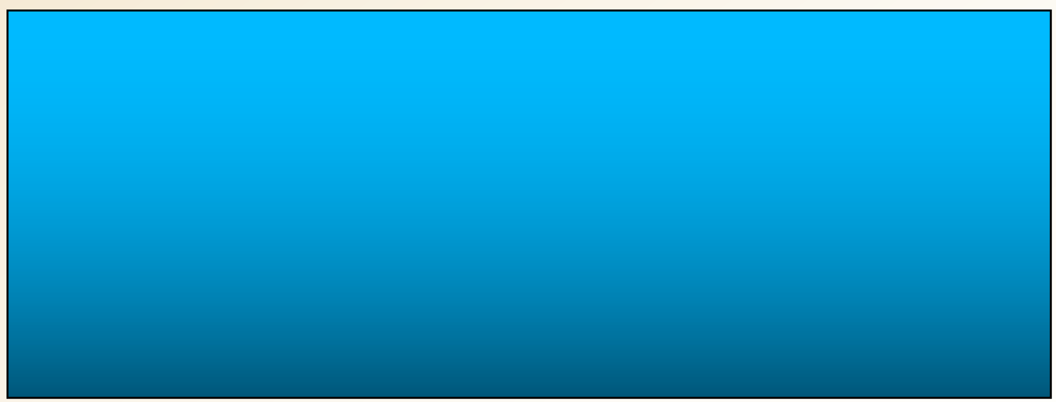


Tamara Rogers, Arizona University

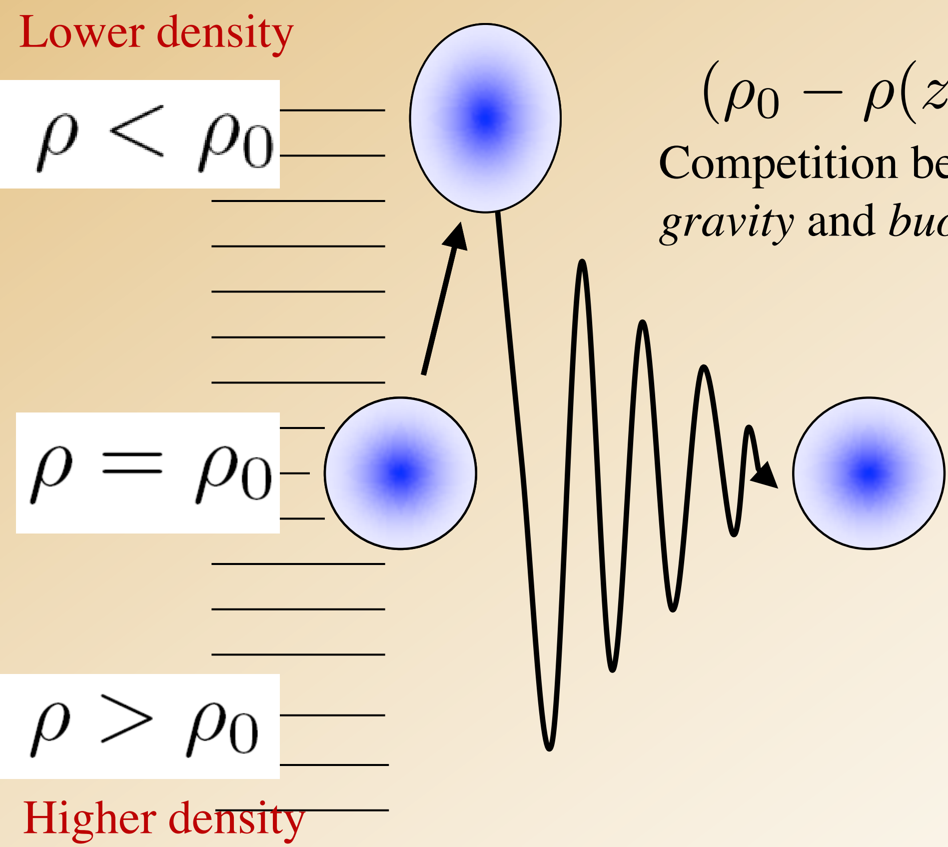
Linear Stratification

Three-layer system

Linear Stratification

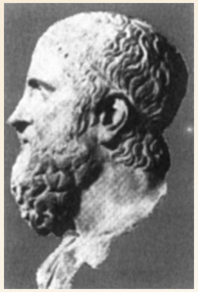


Buoyancy Frequency



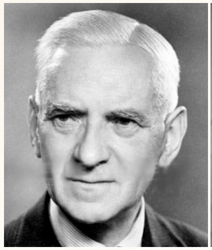
$$(\rho_0 - \rho(z))\,\vec{g}$$

Competition between *gravity* and *buoyancy*



Archimède

$$N^2 = -\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}$$



Brunt

Väisälä

<u>Typical periods</u>

Ocean → 30 min

Laboratory → 10 s

- Slow oscillations
- Wave propagation

Internal Waves

- Internal gravity waves play a primary role in geophysical fluids
  – Significant contribution to mixing in the ocean (Wunsch & Ferrari '04).
  – Redistribution of energy and momentum in the middle atmosphere (Fritts & Alexander '03).

- Mechanisms
  – Generation and propagation are well understood (Garrett & Kunze '07).
  – By contrast, dissipation mechanisms are still debated (Kunze & Llewellyn-Smith '04).
    – Wave-Wave interactions - PSI (MacKinnon & Winters '05, Alford et al. '07).
    – Reflection on sloping boundaries (Nash et al. '04, Dauxois & Young '99).
    – Scattering by mesoscale structures (Rainville & Pinkel '06).
    – Scattering by finite amplitude bathymetry (Johnston & Merrifield & Holloway '03; Peacock, Mercier, Didelle, Viboud, Dauxois '09).
  – Observed energy spectra resulting from NL waves interactions are poorly understood. (Garrett & Munk '75, Nazarenko '11)

Basic equations

Incompressible flow
$$\nabla \cdot \boldsymbol{u} = 0,$$

Navier-Stokes Eq.
$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_{\text{ref}}} \nabla p + b \boldsymbol{e}_z + \nu \nabla^2 \boldsymbol{u}$$

Mass conservation
$$\partial_t b + \boldsymbol{u} \cdot \nabla b + u_z N^2 = 0.$$

$N(z) = \left(-g\left(\partial_z \rho_0\right)/\rho_{\text{ref}}\right)^{1/2}$

$b = g\left(\rho_0(z) - \rho\right)/\rho_{\text{ref}}$

Restricting to 2D and introducing the streamfunction
$$\boldsymbol{u} = (\partial_z \psi, 0, -\partial_x \psi)$$

one gets
$$\partial_t \nabla^2 \psi + J(\nabla^2 \psi, \psi) = -\partial_x b + \nu \nabla^4 \psi.$$
$$\partial_t b + J(b, \psi) - N^2 \partial_x \psi = 0.$$

where $J(\psi, b) = \partial_x \psi\, \partial_z b - \partial_x b\, \partial_z \psi$

One can finally combine above equations in

$$\boxed{\partial_{tt} \nabla^2 \psi + N^2 \partial_{xx} \psi} = \boxed{\nu \nabla^4 \partial_t \psi} + \boxed{\partial_t J(\psi, \nabla^2 \psi) + \partial_x J(b, \psi)}$$

Linear terms | Viscosity | Nonlinear terms

Unusual wave equation: Linear Approximation

$$\nabla^2\psi_{tt} + N^2\psi_{xx} = 0$$

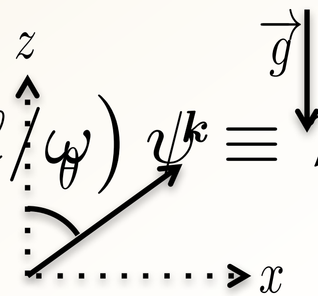
Different from the D'Alembert's equation

Plane wave solution

$$\psi = \psi_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \longrightarrow \omega = \pm N\frac{\ell}{k} = \pm N\sin\theta$$

with $\boldsymbol{k} = (\ell, 0, m)$ and $k = |\boldsymbol{k}| = (\ell^2 + m^2)^{1/2}$

$$\partial_t b = N^2\partial_x\psi \longrightarrow b = -\left(N^2\ell/\omega\right)\psi \equiv \mathcal{P}\psi$$

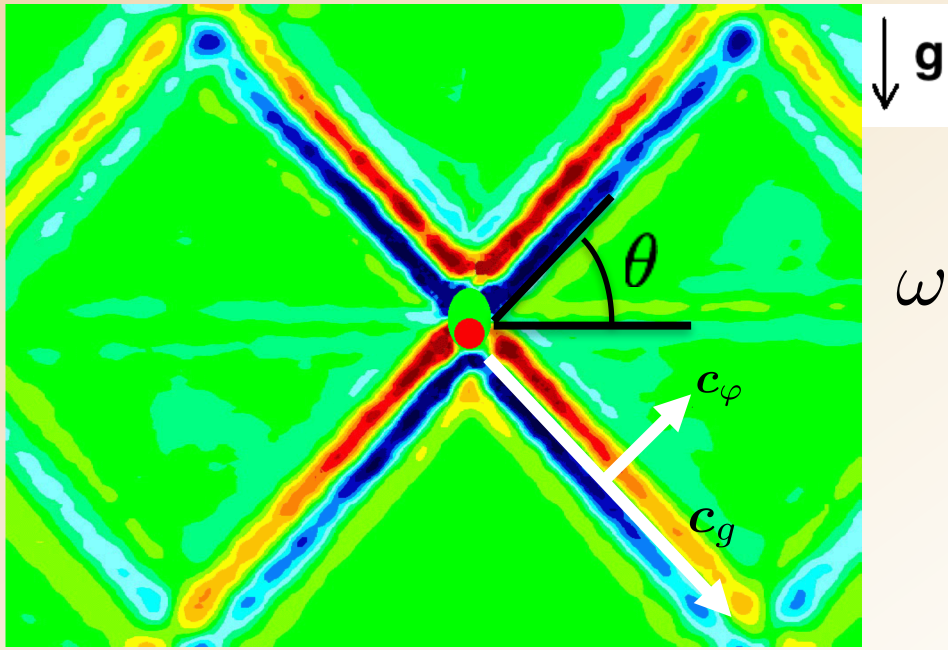


Internal Wave Beams

Usual pedagogical introduction to internal waves: The Saint Andrew's cross

Internal waves emitted by oscillating a cylinder

Lower density



Larger density

$$\omega = \pm N \sin\theta$$

$$N = \sqrt{-\frac{g}{\bar{\rho}}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}}$$

- Anisotropic propagation due to the vertical stratification.
- Dispersion relation features the angle of propagation rather than the wavelength.

E. Ermanyuk

Unusual wave equation: Nonlinear

$$\nabla^2\psi_{tt} + N^2\psi_{xx} = \boxed{\partial_t J(\psi, \nabla^2\psi) + \partial_x J(b, \psi)}$$

Plane wave solution

with $\boldsymbol{k} = (\ell, 0, m)$ and $k = |\boldsymbol{k}| = (\ell^2 + m^2)^{1/2}$

$$\psi = \psi_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \xrightarrow[J(\psi, -k^2\psi) = 0]{} J(\psi, \nabla^2\psi) = 0$$

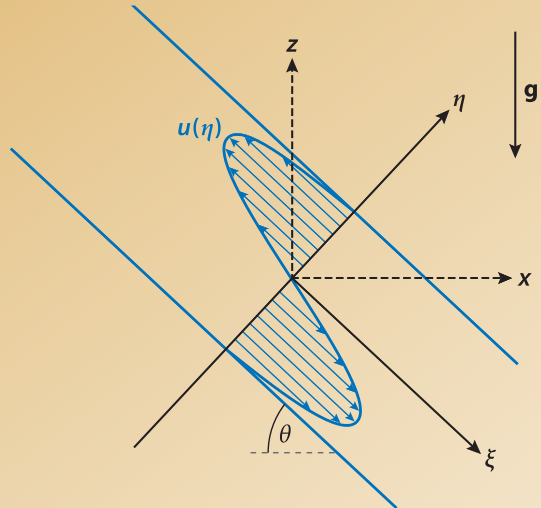
$$b = -\left(N^2\ell/\omega\right)\psi \equiv \mathcal{P}\psi \xrightarrow[J(\psi, \mathcal{P}\psi) = 0.]{} J(\psi, b) = 0$$

$$\boxed{\partial_{tt}\nabla^2\psi + N^2\partial_{xx}\psi} = \boxed{\partial_t J(\psi, \nabla^2\psi) + \partial_x J(b, \psi)} = 0$$

Plane waves are solutions of the *nonlinear* equation:
even for large amplitude

Internal Wave Beams

McEwan (1973), Tabaei, Akylas & Lamb (2005)



$$\boldsymbol{u} = u(\eta)(\cos\theta, 0, -\sin\theta)e^{-i\omega t} + c.c.$$
$$b = -i(\mathcal{P}/k)u(\eta)e^{-i\omega t} + c.c.\,.$$

with an arbitrary complex amplitude $u(\eta)$

$\boldsymbol{u}$ and b do not depend on the longitudinal variable ξ $\longrightarrow$ $J(\psi, b) = \partial_\xi\psi\,\partial_\eta b - \partial_\xi b\,\partial_\eta\psi = 0$

$J(\psi, \nabla^2\psi) = 0$

$$\boxed{\partial_{tt}\nabla^2\psi + N^2\partial_{xx}\psi} = \boxed{\partial_t J(\psi, \nabla^2\psi) + \partial_x J(b, \psi)} = 0$$

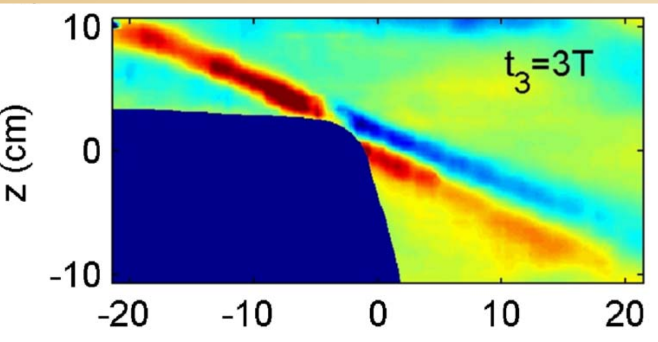
A uniform (along ξ) beam, regardless of its profile (along η), is an exact NL solution.

- Prolongations: Nonlinear terms cancel out when considering
  - Reflection of Internal Waves: Dauxois & Young, *JFM* (1999)
  - Modulated Nonlinear Beams: Tabaei & Akylas, *JFM* (2003)

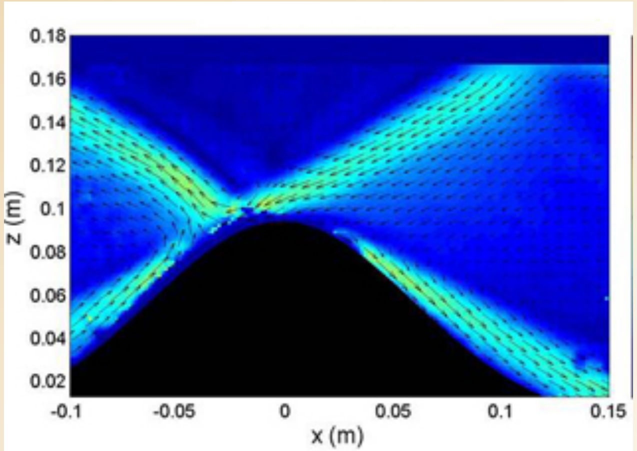
- Consequences:
  - Nonlinearity has only relative weak consequences…at first sight.
  - Applicability of linear results to field observations, lab. expts or numerical simulations.

Tidal flow over topography

Experiments

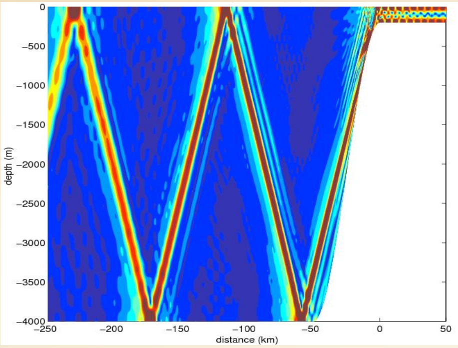


Gostiaux & Dauxois, PoF 2007

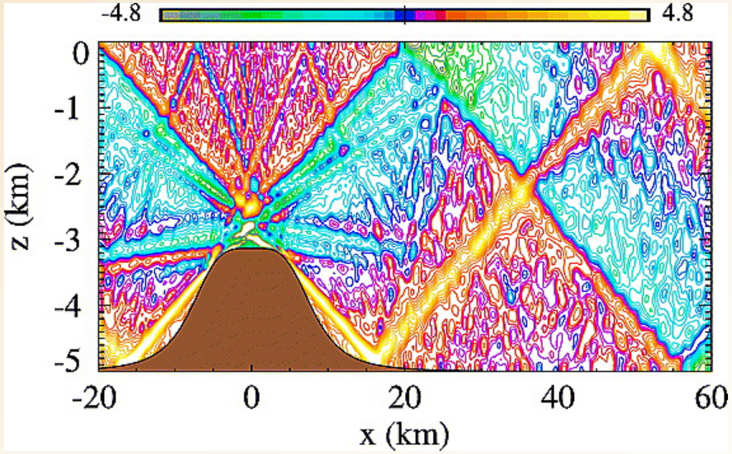


Peacock, Echeverri & Balmforth, JPO 2008

Numerical Simulations

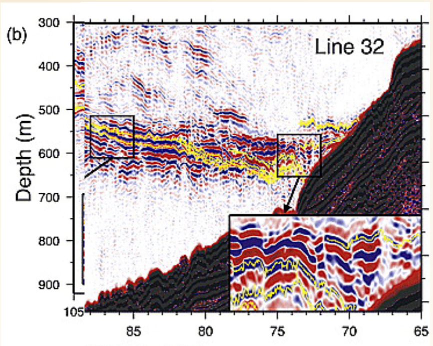


Maugé & Gerkema, NPG 2008

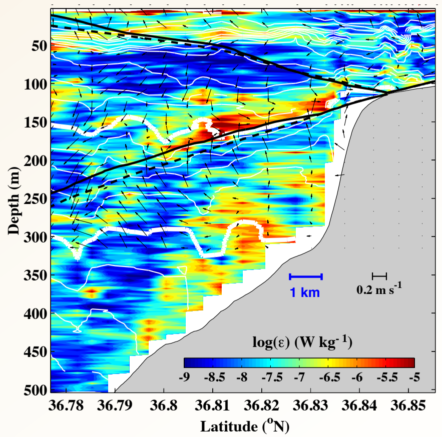


Lamb, GRL 2004

Fields observations



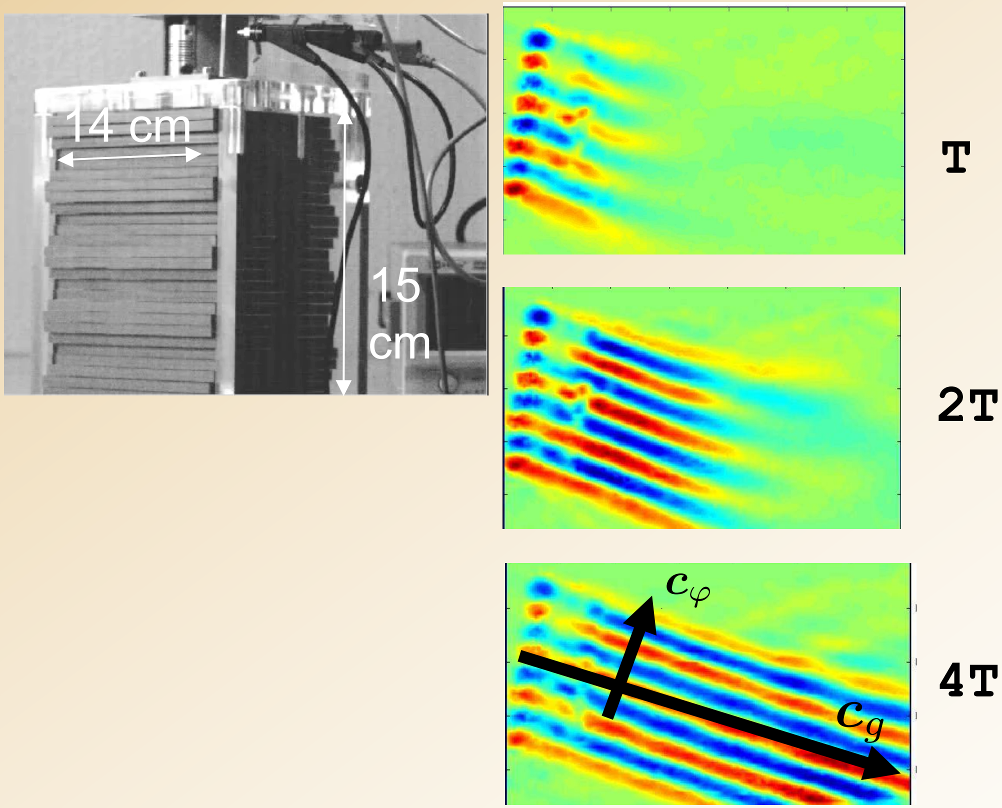
Holbrok & Fer, GRL 2005



Lien & Gregg, JGR 2001

Experimental Internal Wave Beams

-Gostiaux, Didelle, Mercier, Dauxois *Exp. in Fluids* (2007)
-Mercier, Mathur, Gostiaux, Martinand, Peacock, Dauxois *JFM* (2010)



Advantages: -Well defined beam
-Only *one* beam
-Wavelength < Width
-Emission localized in space

Résumé

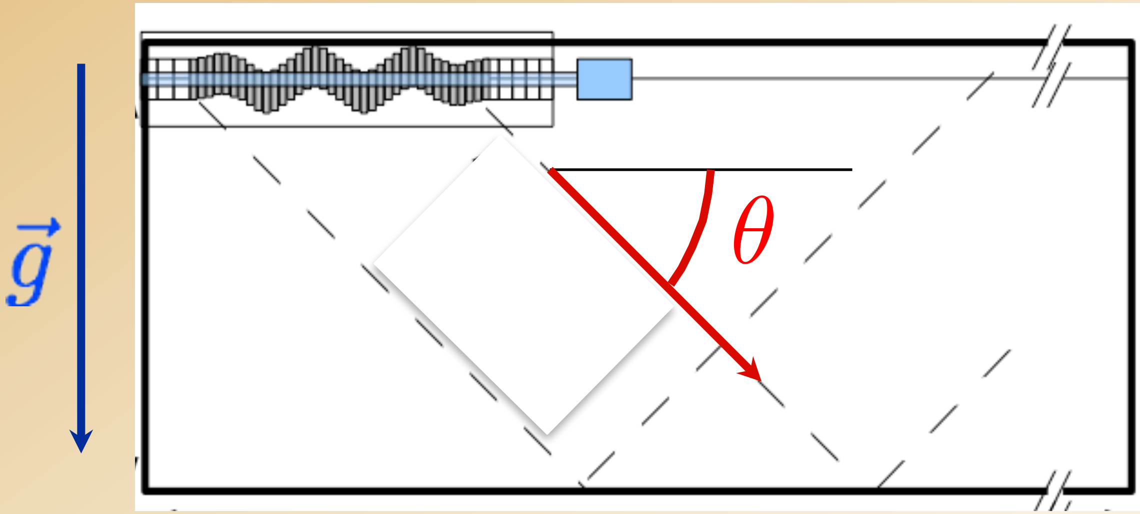
- State:
  - Internal wave beams are solutions of the Nonlinear equations
  - Identifying a solution does not mean that it is a stable one!

- Objective of this talk:
  - Present recent progress on NL destabilization of *internal wave beams*
  - Bridging part of the gap between our understanding of their
    - Generation mechanisms based mostly on linear analysis,
    - subsequent evolution through nonlinear effects.
  - Studying Abyssal mixing in the lab. with internal wave beams.

- Dauxois, Joubaud, Odier, Venaille, Annual Review of Fluid Mechanics 2018

Stability of Internal Waves

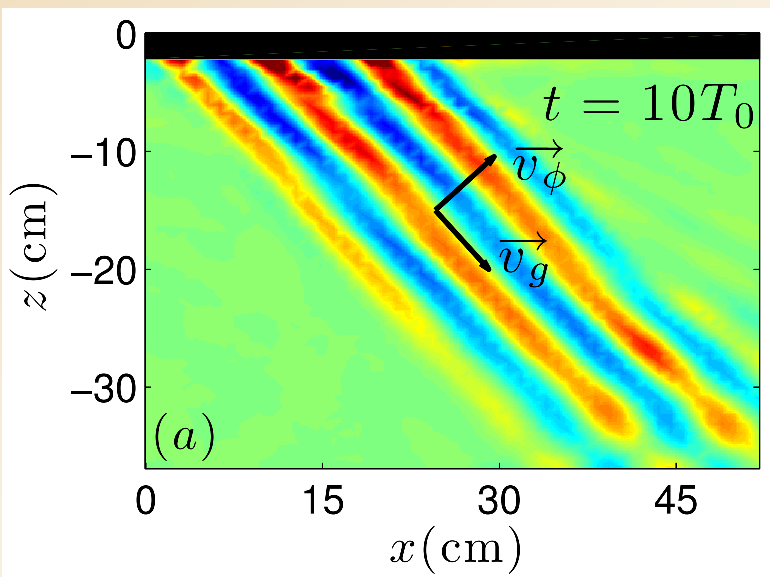
Linear Internal Waves



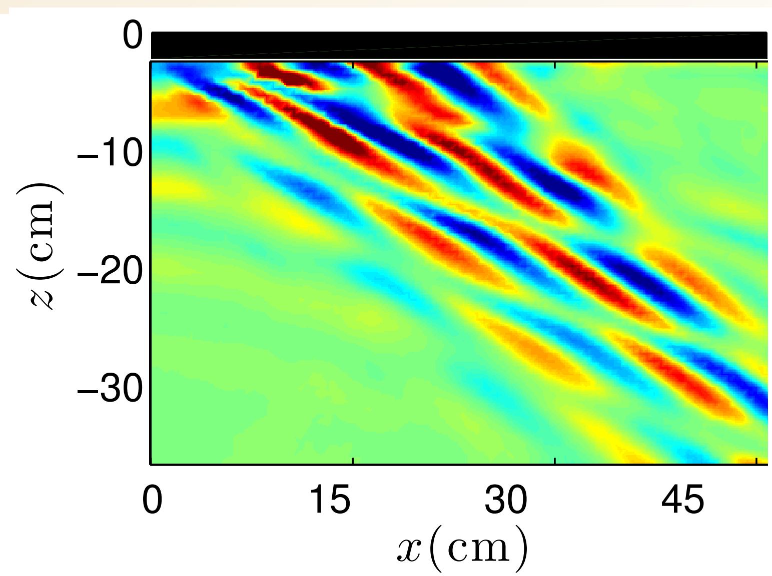
Dispersion relation

$$\omega = \pm N \frac{\ell}{\sqrt{\ell^2 + m^2}} = \pm N \sin\theta$$

$$N = \sqrt{-\frac{g}{\bar{\rho}}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}}$$

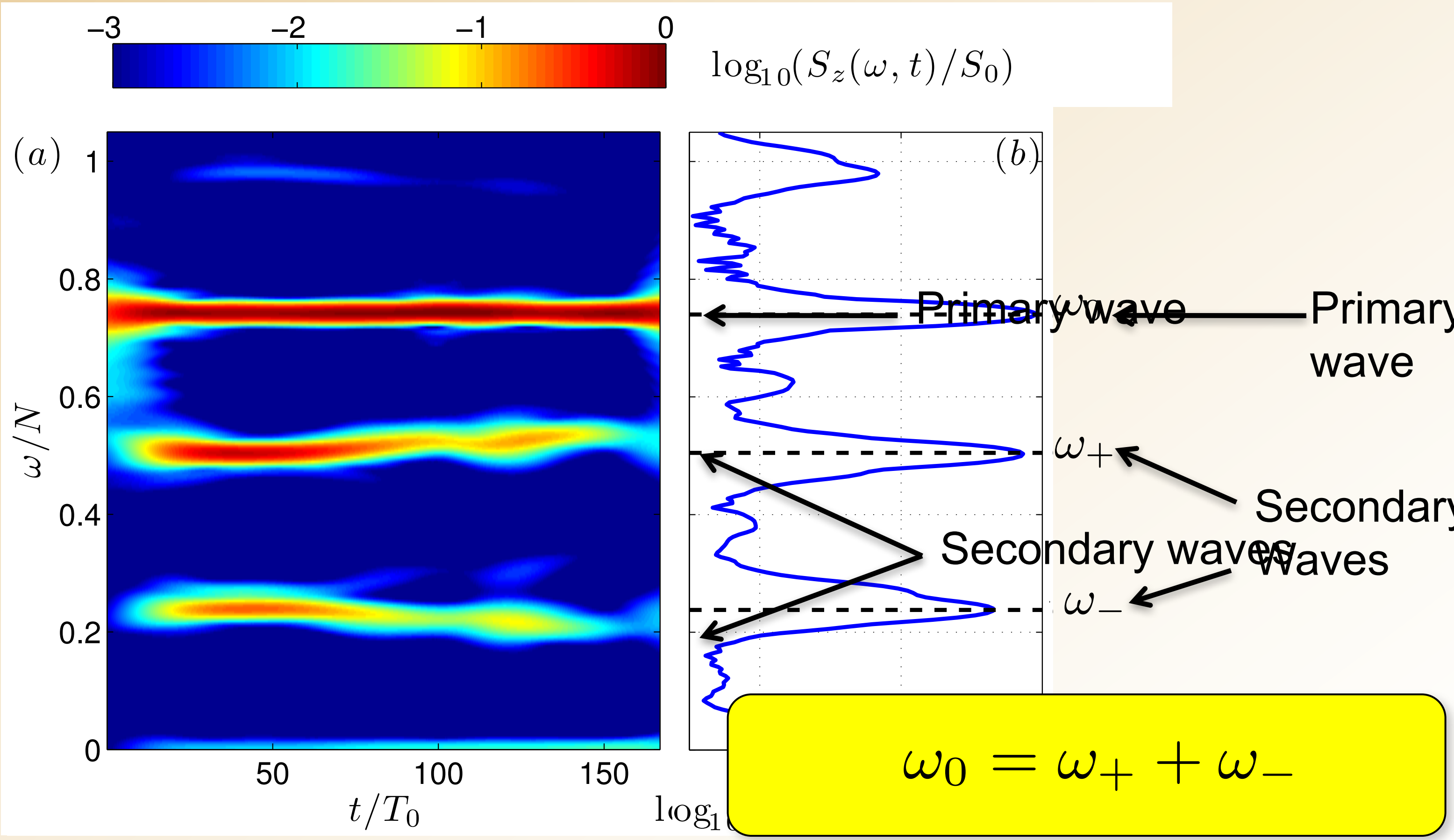


Stable



Unstable

Time-frequency Analysis

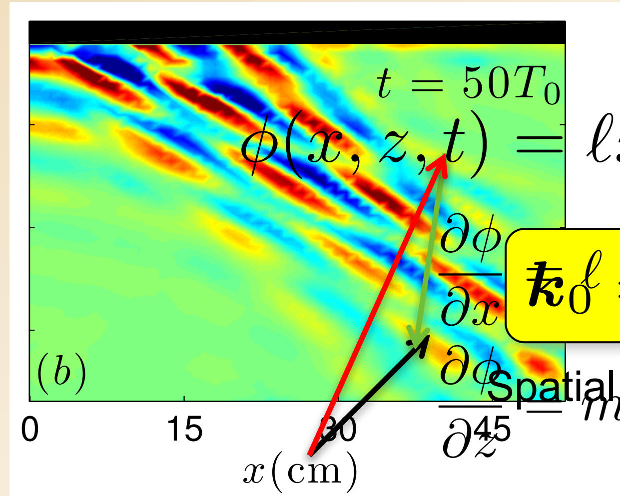


$$\omega_0 = \omega_+ + \omega_-$$

Temporal resonance condition

$$S_r(\omega,t) = \left\langle \left| \int_{-\infty}^{+\infty} \mathrm{d}u\, \tilde{\rho}_r(u)\, e^{i\omega u}\, h(t-u) \right|^2 \right\rangle_{xz}$$

Sample Experiment



$$\phi(x,z,t) = \ell x + mz - \omega t$$

$$\frac{\partial\phi}{\partial x} = \ell \qquad \frac{\partial\phi}{\partial z} = m$$

$$\boldsymbol{k} = (\ell, 0, m)$$

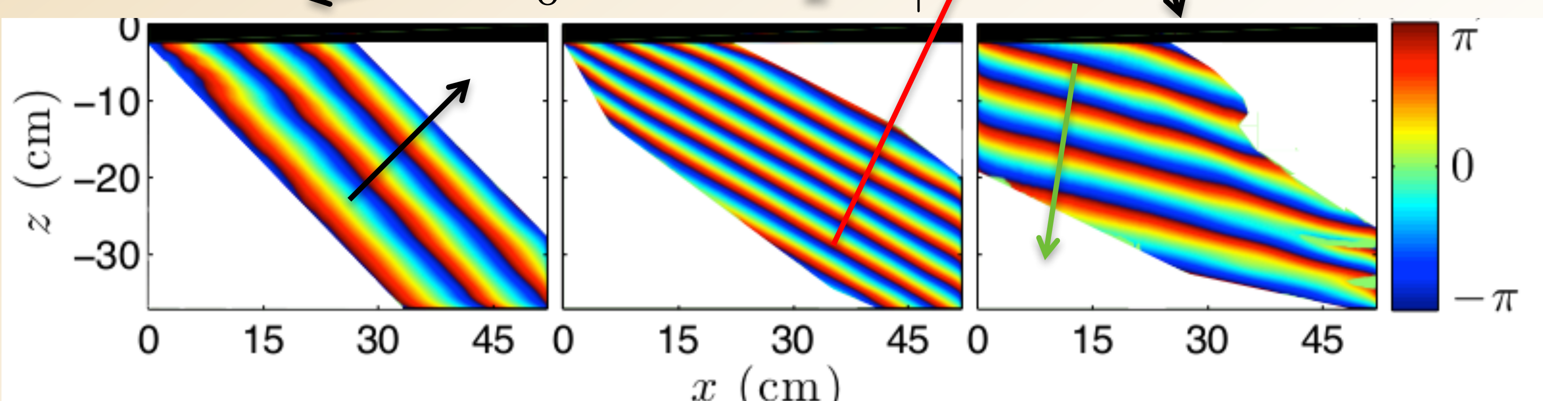
$$\boldsymbol{k}_0 = \boldsymbol{k}_+ + \boldsymbol{k}_-$$

Spatial resonance condition

Mercier, Garnier, Dauxois, PoF **20**, 086615 (2008)

Hilbert Transform

ω_0 ω_+ ω_-



Different frequencies imply different angle of propagation

Measurement of the phase of each component using the Hilbert transform

Theory of this triadic instability

McEwan & Plumb, DAO 2, 83 (1977).

Koudella & Staquet JFM 548, 165 (2006).
Bourget, Dauxois, Joubaud, Odier, JFM 723, 1 (2013).

$$\partial_t \nabla^2 \psi + J(\nabla^2 \psi, \psi) = -\partial_x b + \nu \nabla^4 \psi.$$

$$\partial_t b + J(b, \psi) - N^2 \partial_x \psi = 0.$$

Seeking solutions of the form

$$b = \sum_j R_j(t) e^{i(\boldsymbol{k}_j \cdot \boldsymbol{r} - \omega_j t)} + c.c.$$

$$\psi = \sum_j \Psi_j(t) e^{i(\boldsymbol{k}_j \cdot \boldsymbol{r} - \omega_j t)} + c.c.,$$

We find that provided the spatial $\boldsymbol{k}_0 = \boldsymbol{k}_+ + \boldsymbol{k}_-$ and temporal $\omega_0 = \omega_+ + \omega_-$ resonance conditions are satisfied, we get

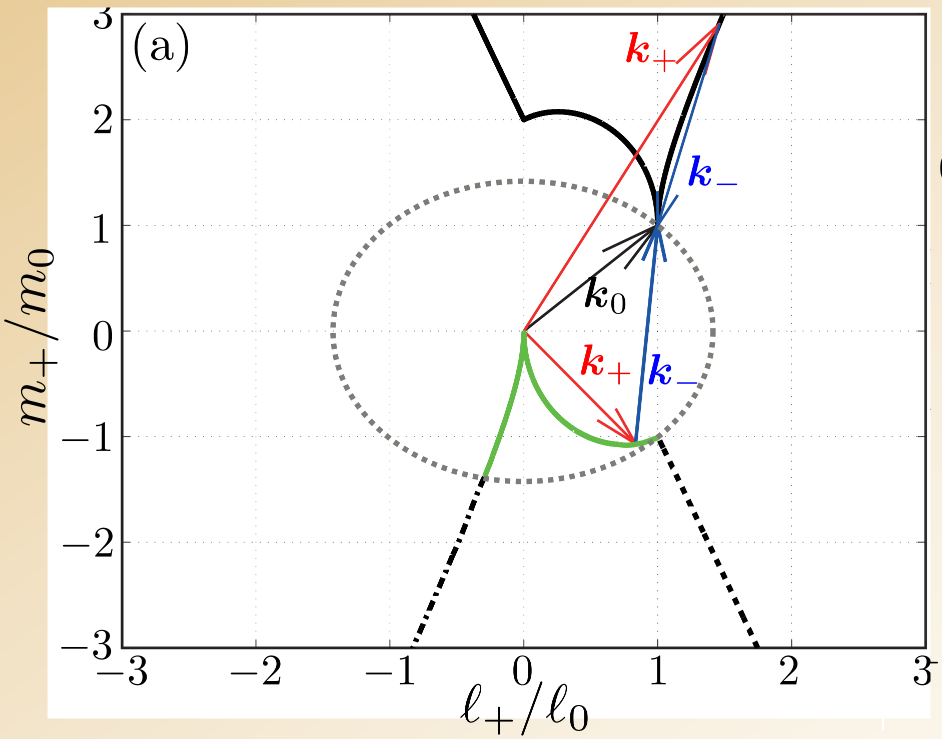
$$\frac{\mathrm{d}\Psi_\pm}{\mathrm{d}t} = |I_\pm| \Psi_0 \Psi_\mp^* - \frac{\nu}{2} k_\pm^2 \Psi_\pm$$

where $I_\pm = (\ell_0 m_\mp - m_0 \ell_\mp)[\omega_\pm (k_0^2 - k_\mp^2) + \ell_\pm N^2 (\ell_0/\omega_0 - \ell_\mp/\omega_\mp)]/(2\omega_\pm k_\pm^2)$.

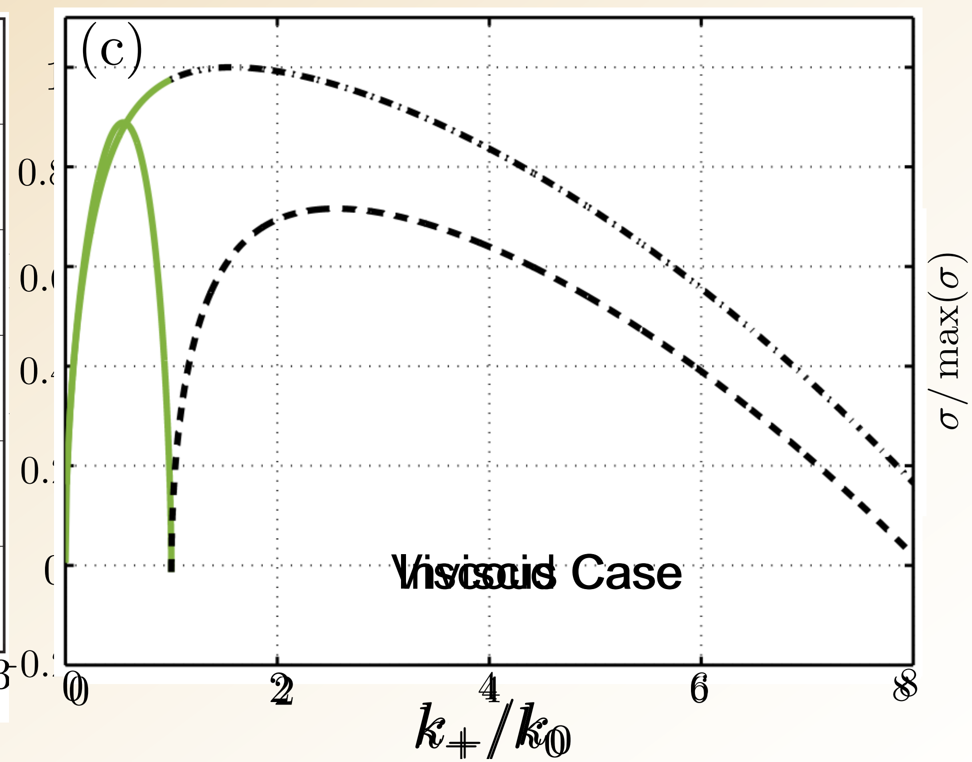
Resonant triads

$$\omega_0 = \omega_+ + \omega_- \quad \text{and} \quad \boldsymbol{k}_0 = \boldsymbol{k}_+ + \boldsymbol{k}_-$$

Resonance locus



Growth rate



1. Good agreement between theory and experiments

-Parametric Subharmonic Instability (PSI): amplification of short-scale perturbations with frequency equal to one half of that of the underlying wave ➔ Nearly inviscid case

2. However one never attains the central curves although they have larger growth rates.

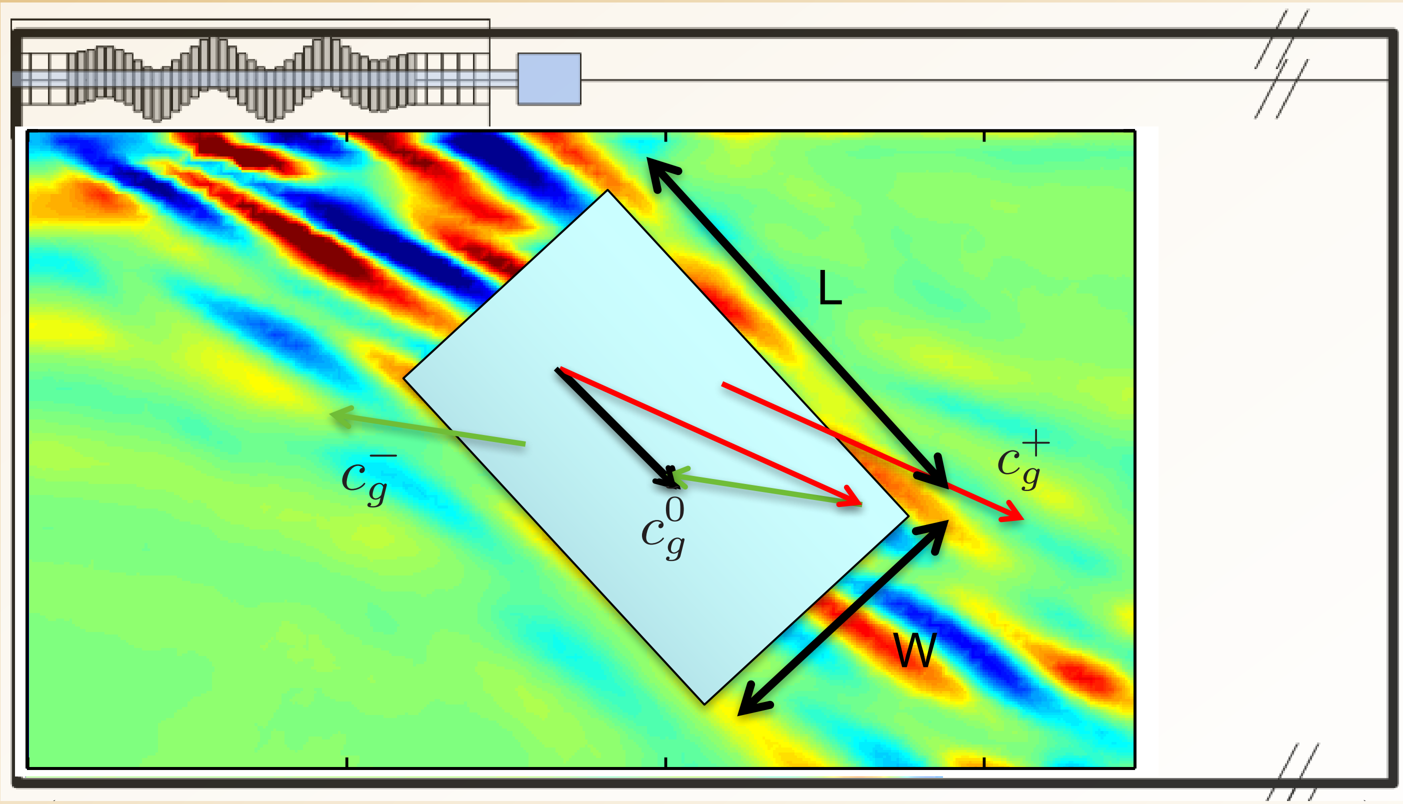
Why?

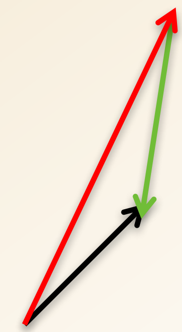
No threshold for an internal *plane* wave.

-Triadic Resonant Instability (TRI) is a more general denomination

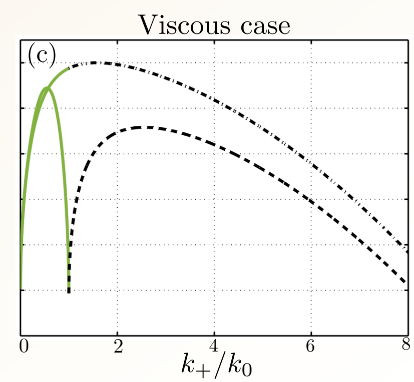
Bourget, Scolan, Dauxois, Le Bars, Odier, Joubaud, *JFM* 759, 739 (2014).

Finite Width effects





$$\boldsymbol{k}_0 = \boldsymbol{k}_+ + \boldsymbol{k}_-$$



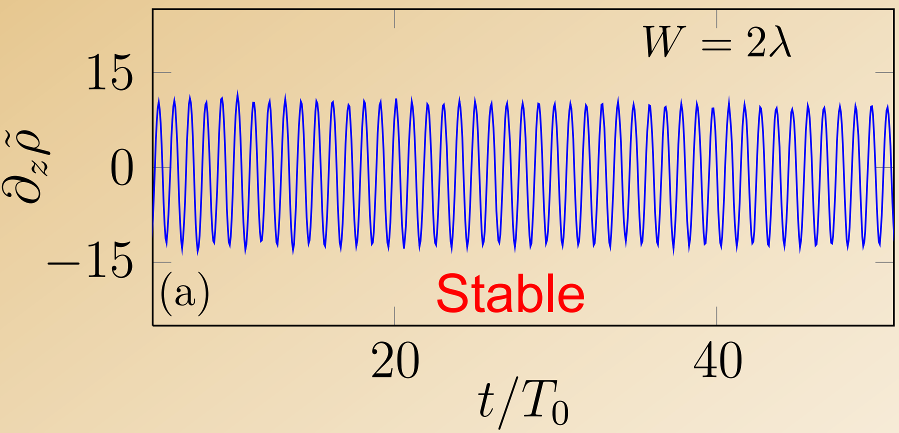
$$c_{g,\pm} = (N^2 - \omega_\pm^2)^{1/2}/k_\pm$$

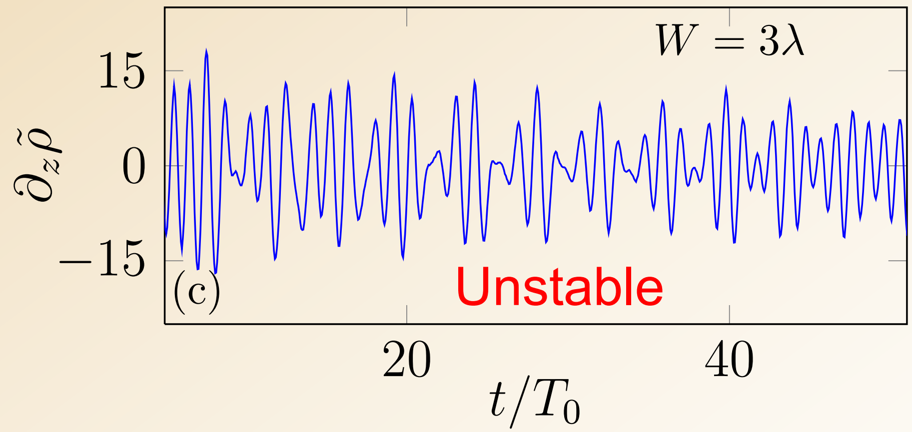
Secondary waves with small wave vectors (having a larger group velocity), leave the primary wave beam more rapidly

Bourget, Scolan, Dauxois, Le Bars, Odier, Joubaud, *JFM* 759, 739 (2014)

Finite Width effects

Experimental Results





Thinner beams are more stable than expected!

Energy approach

Secondary waves $\Rightarrow$

$$\frac{\mathrm{d}\Psi_\pm}{\mathrm{d}t} = |I_\pm|\Psi_0\Psi_\mp^* - \frac{\nu}{2}k_\pm^2\Psi_\pm - \frac{|\boldsymbol{c}_{g,\pm}\cdot \boldsymbol{e}_{k_0}|}{2W}\Psi_\pm$$

Viscosity Beam width

$W \gg \lambda_0$ $\Rightarrow$ Identical to the plane wave case.

$W \simeq \lambda_0$ $\Rightarrow$ The instability is significantly reduced. Strong ``dissipation''.

Prolongations

- Theory: Theoretical prediction: Karimi & Akylas JFM (2014).

- Numerical simulations: Absolute and convective instability:
  - Lerisson's PhD thesis (2017)
  - Lerisson & Chomaz PR Fluids (2018)

- *E pur si muove!* (and yet it moves!): $c_g \sim \sqrt{(\omega^2 - f^2)(N^2 - \omega^2)}/(\omega k)$

  Rotation reducing the ability of subharmonic waves to escape,
  => it reinforces the instability

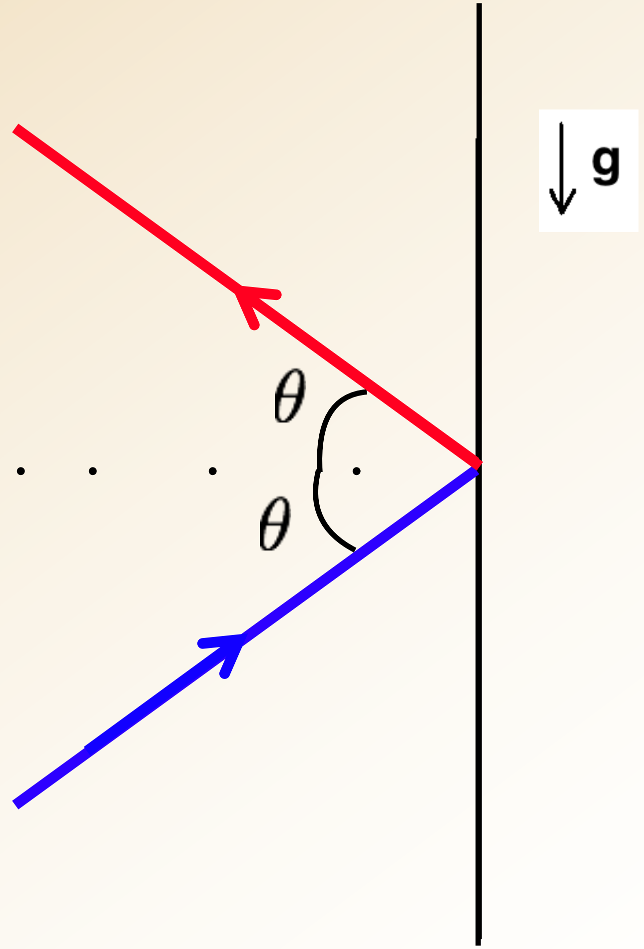
  - Experiments: Maurer, Joubaud & Odier JFM (2016).
  - Theory: Karimi & Akylas PR Fluids (2017).

Energy cascade in internal wave attractors

Internal Waves Reflection

Dispersion relation: $\Omega = \frac{\omega}{N} = \pm \sin\theta$

Reflection on a *vertical* wall



1) for a *ray*

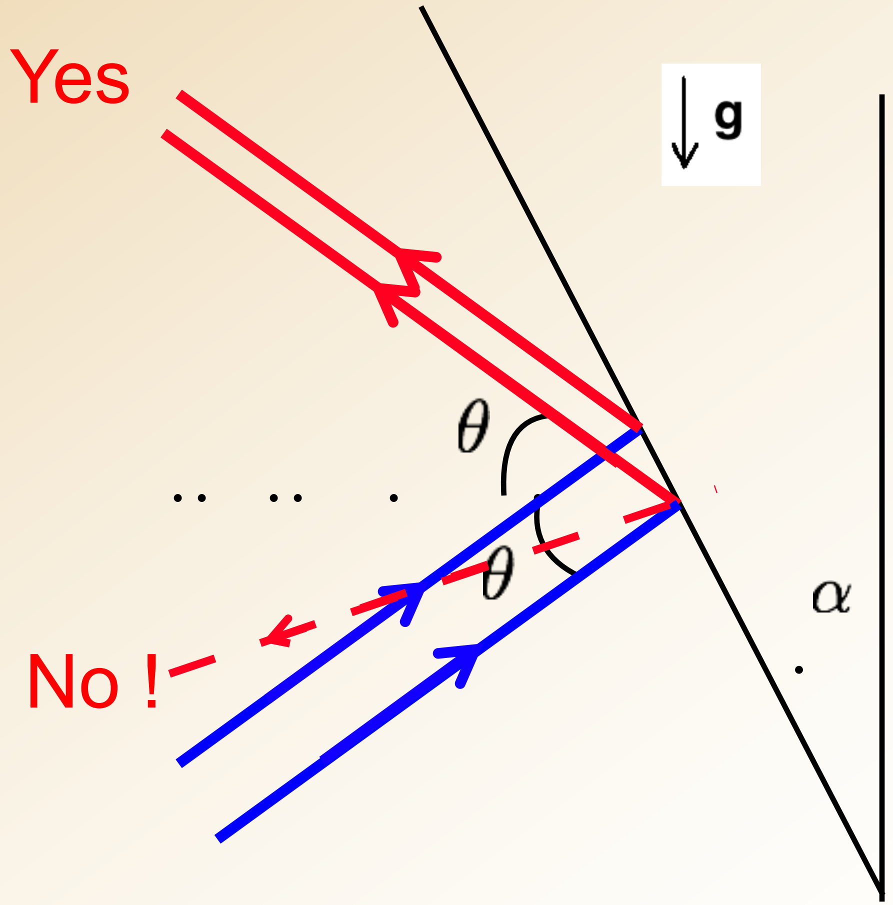
Analogous to the « Snell-Descartes » reflection

Internal Waves Reflection

Dispersion relation: $\Omega = \dfrac{\omega}{N} = \pm \sin\theta$

Reflection on an *inclined* wall

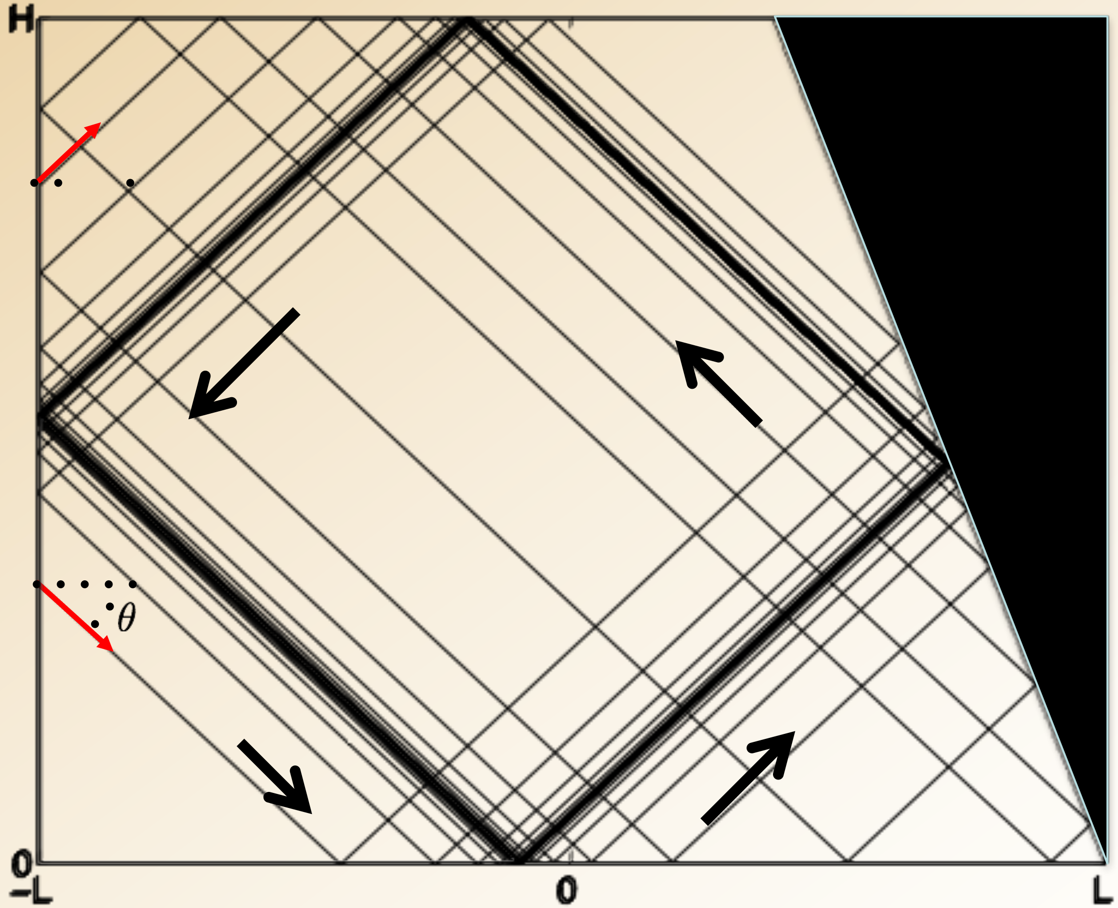
2) For a *beam*



Energy focusing: Linear transfer to small scales

Internal Wave Attractor

Theoretical prediction



Maas

Internal wave attractor which corresponds to the existence of a *limit cycle*, depending on the geometrical parameters (Maas *et al.*, Nature 1997).

Internal Wave Attractor

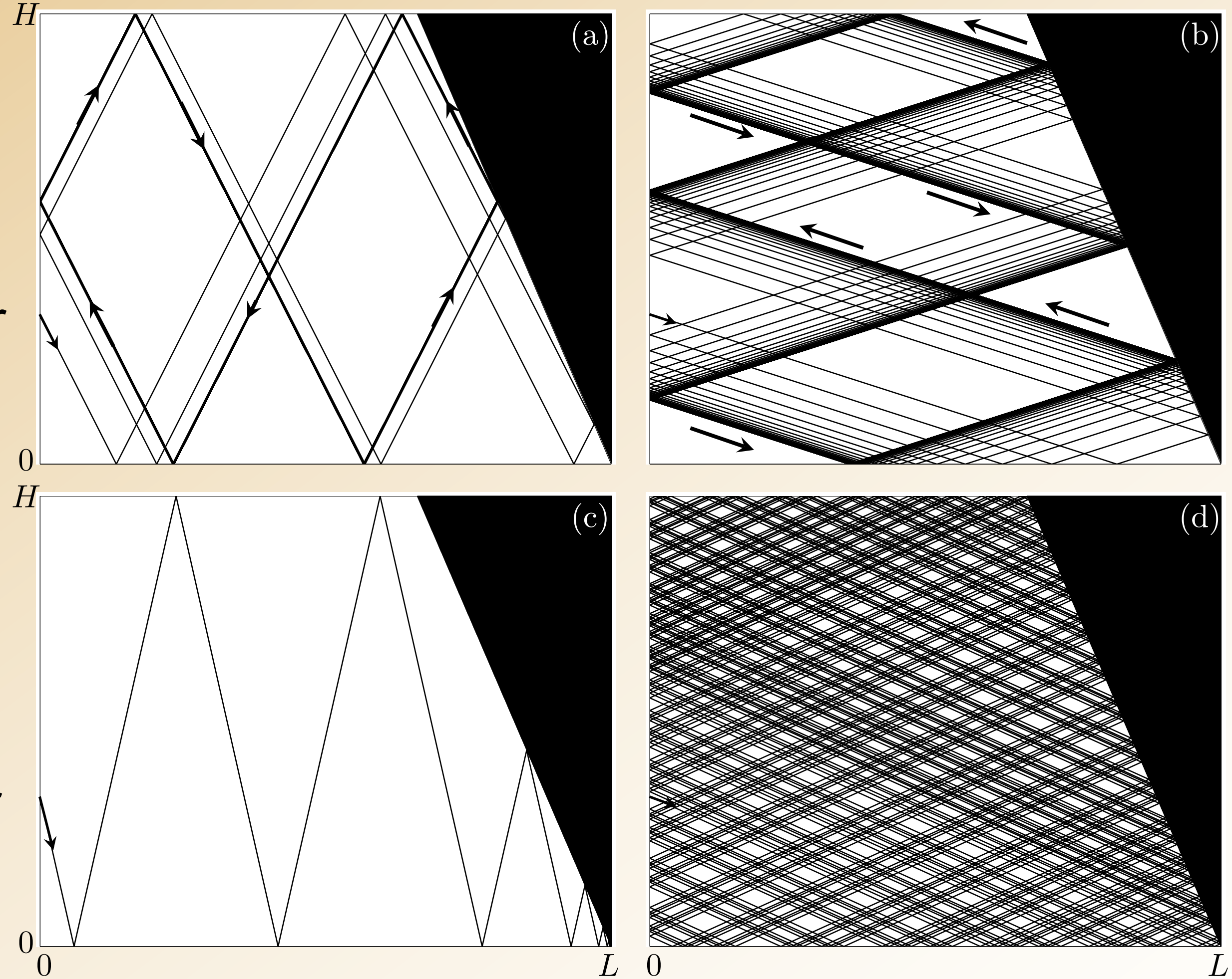
More complex attractors

$\theta = 58^\circ$

(2,1) attractor

$\theta = 75^\circ$

Point attractor



$\theta = 15^\circ$

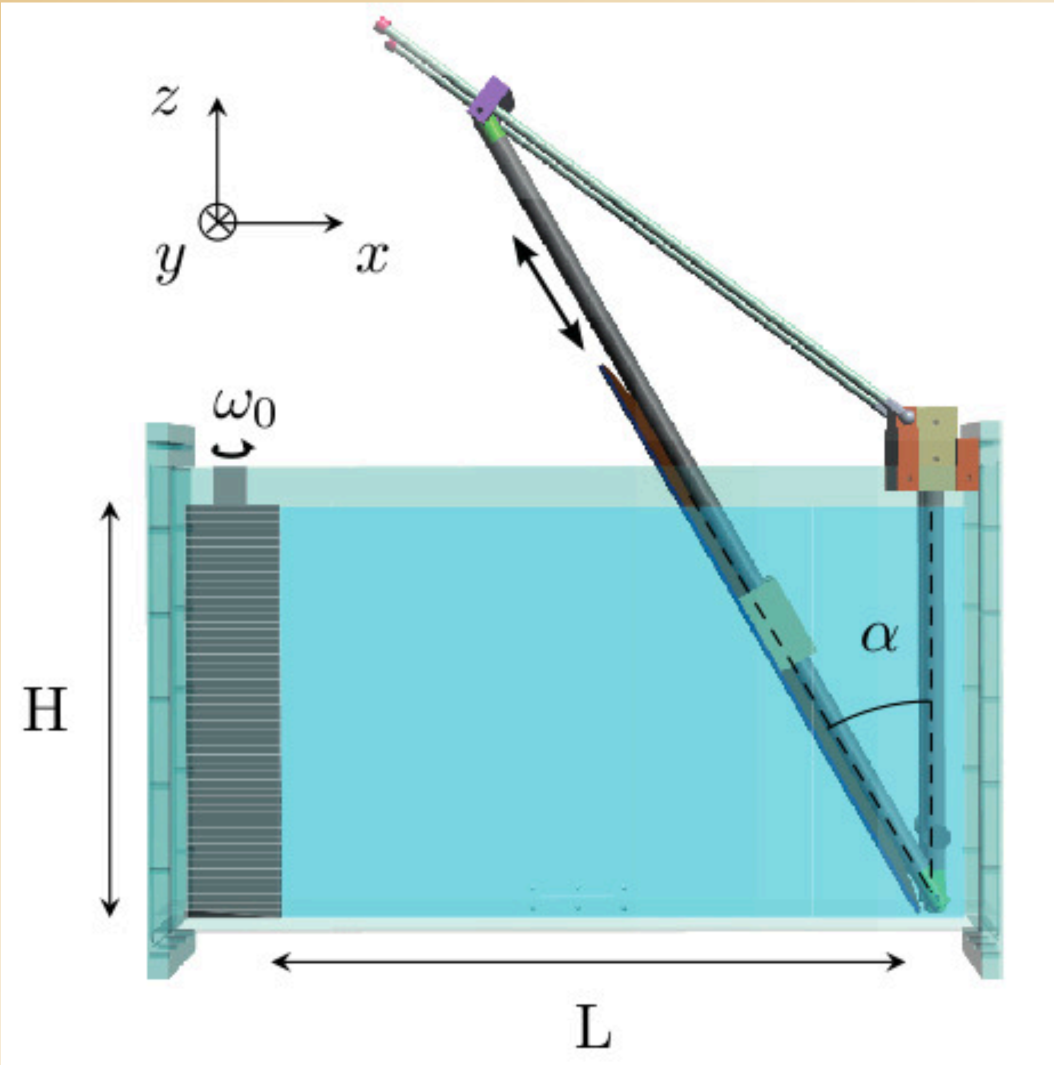
(1,3) attractor

$\theta = 21^\circ$

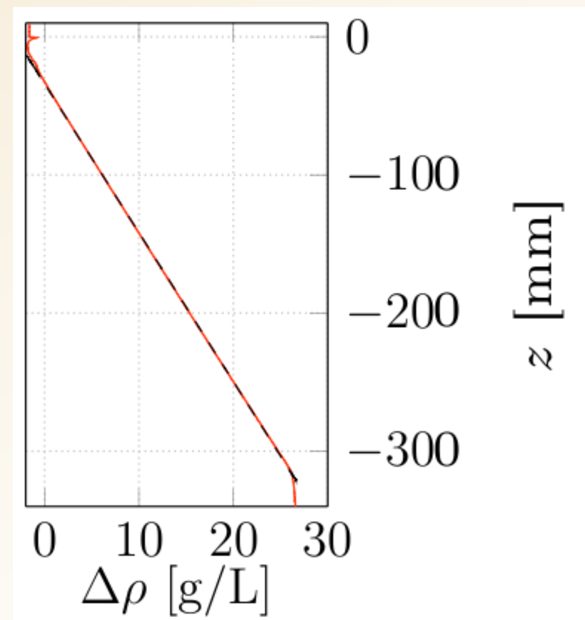
No attractor

An internal wave billiard

Experimental setup:



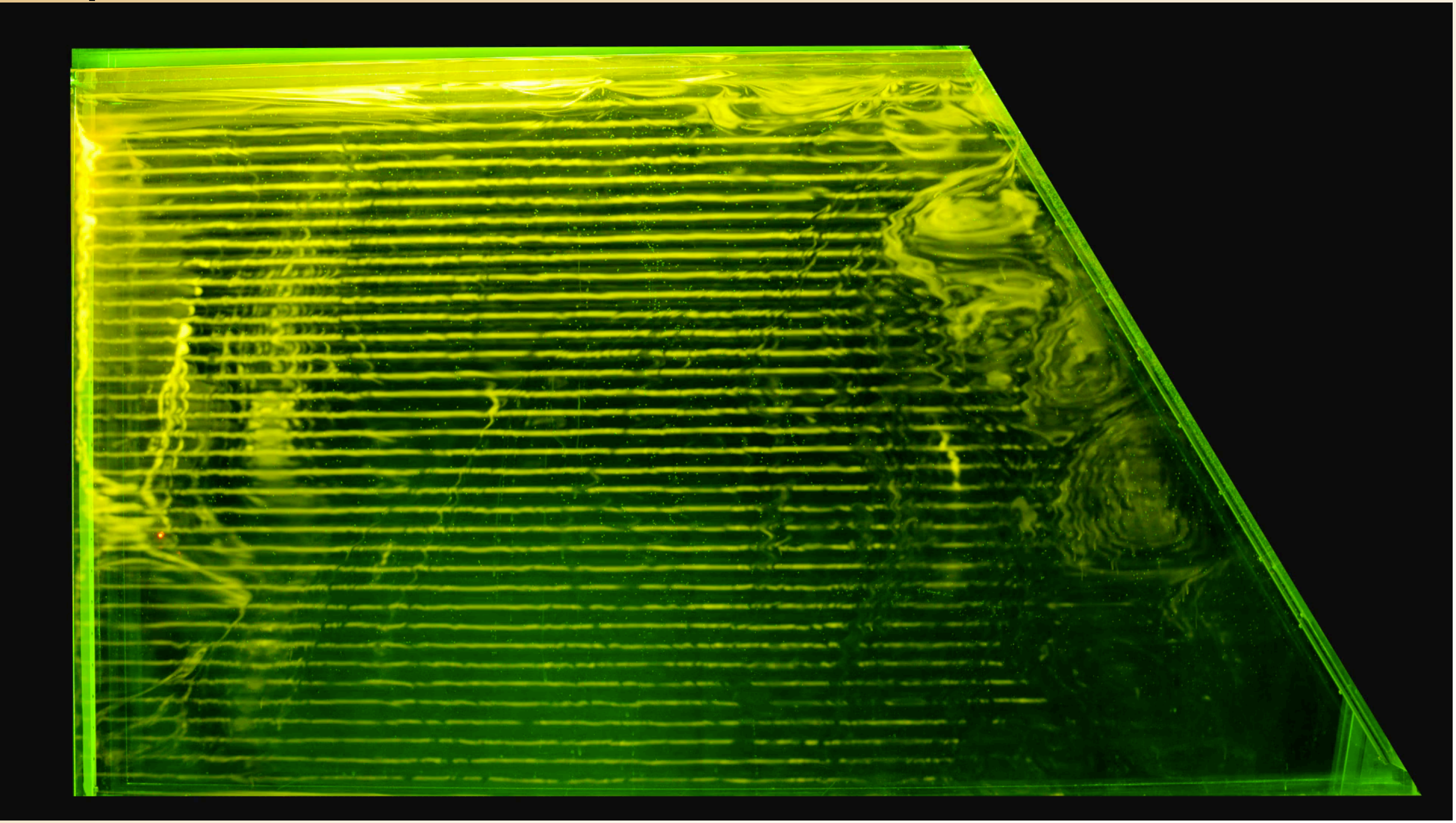
2D flow



Generator profile: $\eta(z,t) = a\cos(\pi z/H)\cos(\omega t)$

Internal Wave Attractor

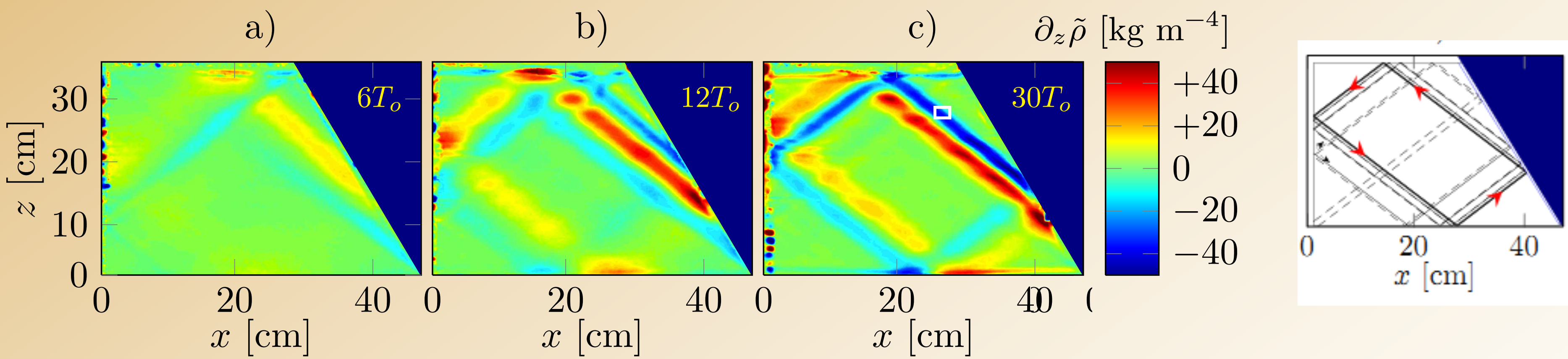
Experimental result:




Davis

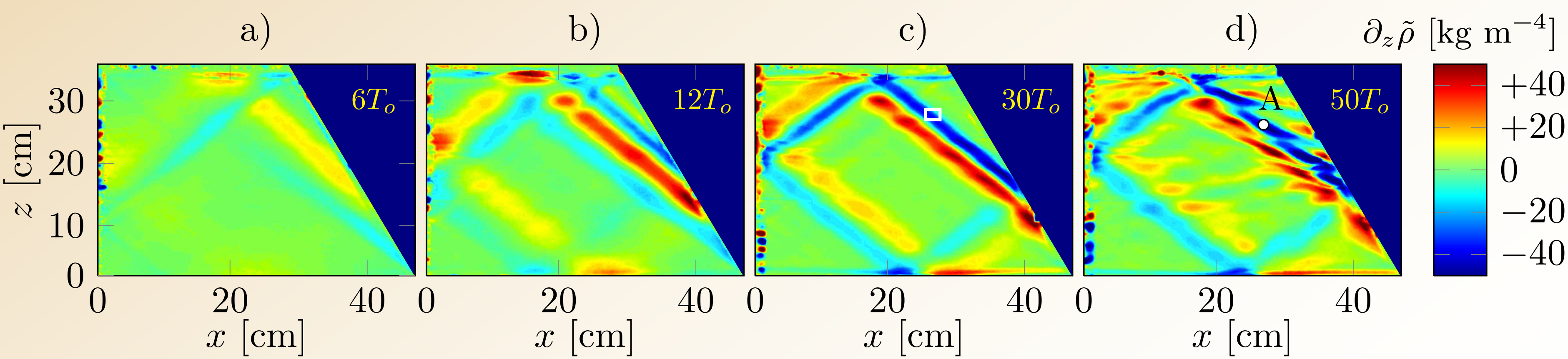

Paillat

Experimental Results



Scolan, Ermanuyk, Dauxois, *Phys. Rev. Lett. 110*, 234501 (2013).

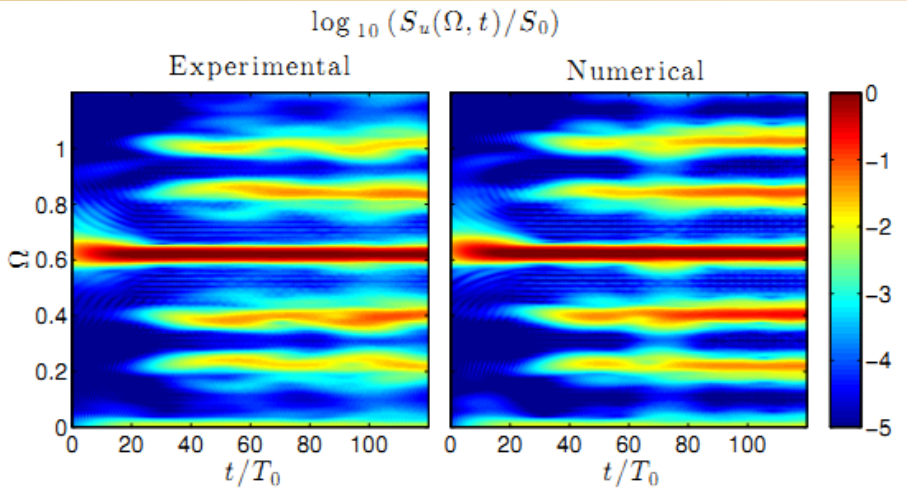
Balance between viscous damping and focusing.

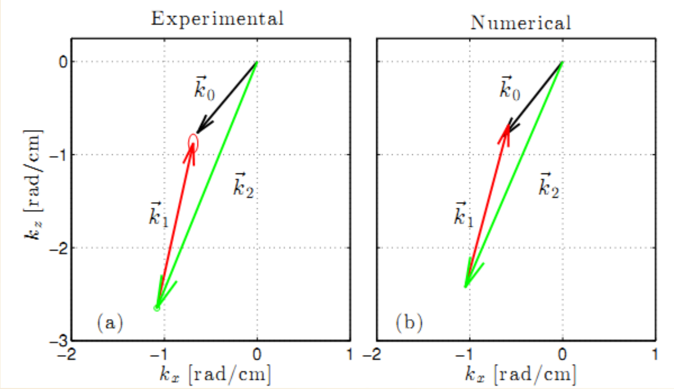


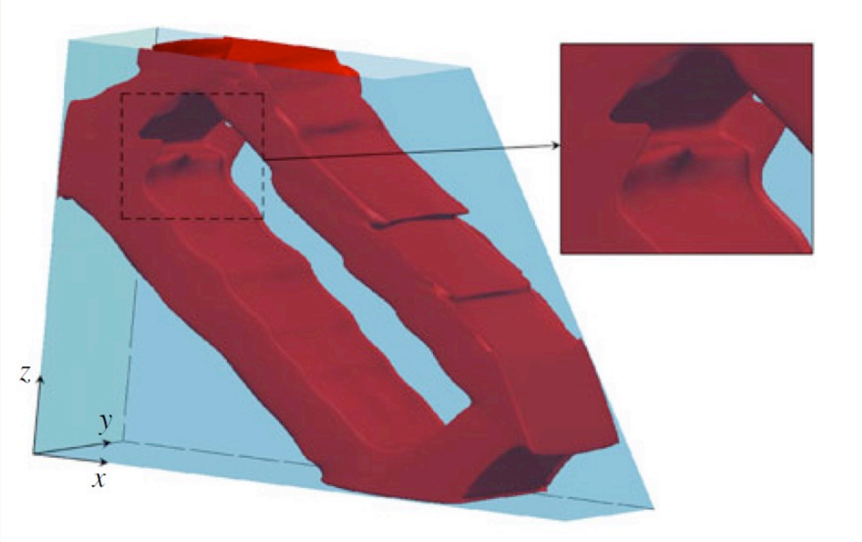
Instability in internal wave attractors for large forcing

Numerical Calculations

Model: Navier-Stokes in Boussinesq approximation + continuity + salt transport
Method: spectral elements 2D and 3D, code Nek5000 (Fischer & Ronquist 1994)
BC: no-slip at rigid walls, stress-free at free surface





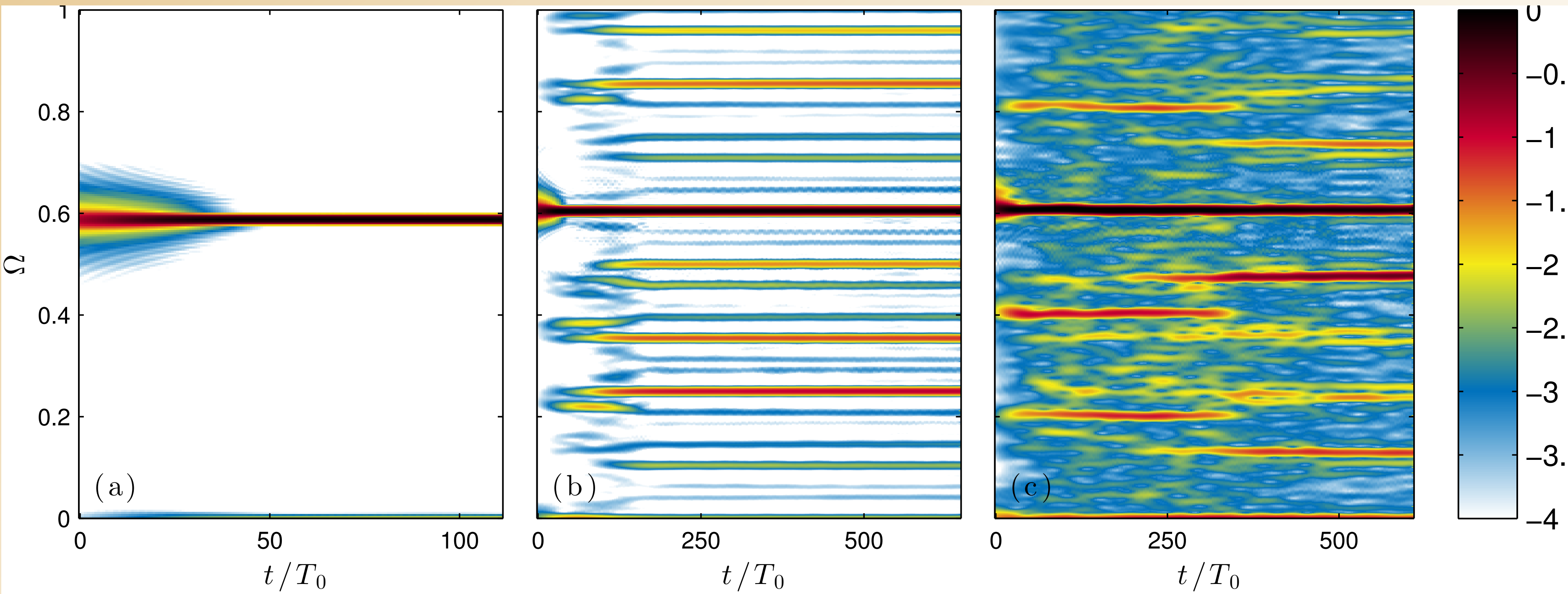


Brouzet, Sibgatullin, Scolan, Ermanyuk, Dauxois JFM (2016)
Internal wave attractors examined using laboratory experiments and 3D numerical simulations

Energy Cascade

Time-frequency diagrams: $S_u(\Omega,t) = \left\langle \left| \int_{-\infty}^{+\infty} u(x,z,\tau) e^{i\Omega N\tau} h(t-\tau)\, d\tau \right|^2 \right\rangle_{xz}$

Low Forcing | Intermediate Forcing | Large Forcing



attractor amplitude

Energy Cascade

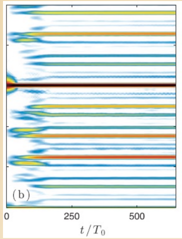
Brouzet, Ermanyuk, Joubaud, Sibgatullin, Dauxois, *EPL* 113, 44001 (2016)

TRI portrayed by the Bicoherence

Bicoherence $M(\Omega_1, \Omega_2) = F(\Omega_1)F(\Omega_2)F^*(\Omega_1 + \Omega_2)$

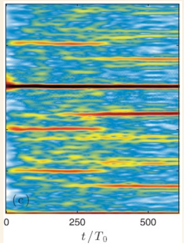
F Fourier transform

Cascade B

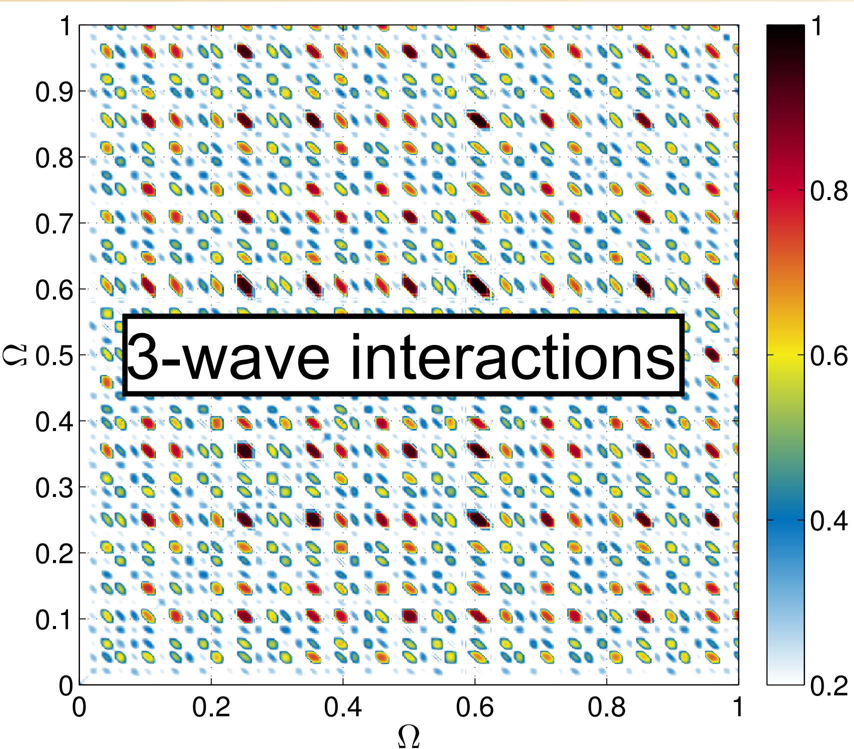


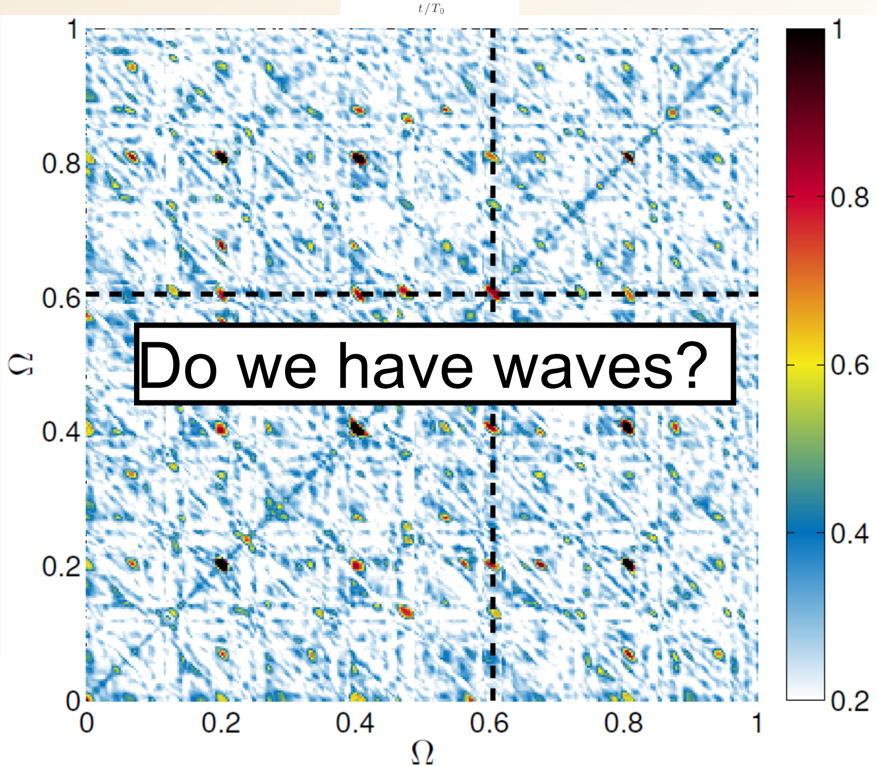
Intermediate forcing

Cascade C

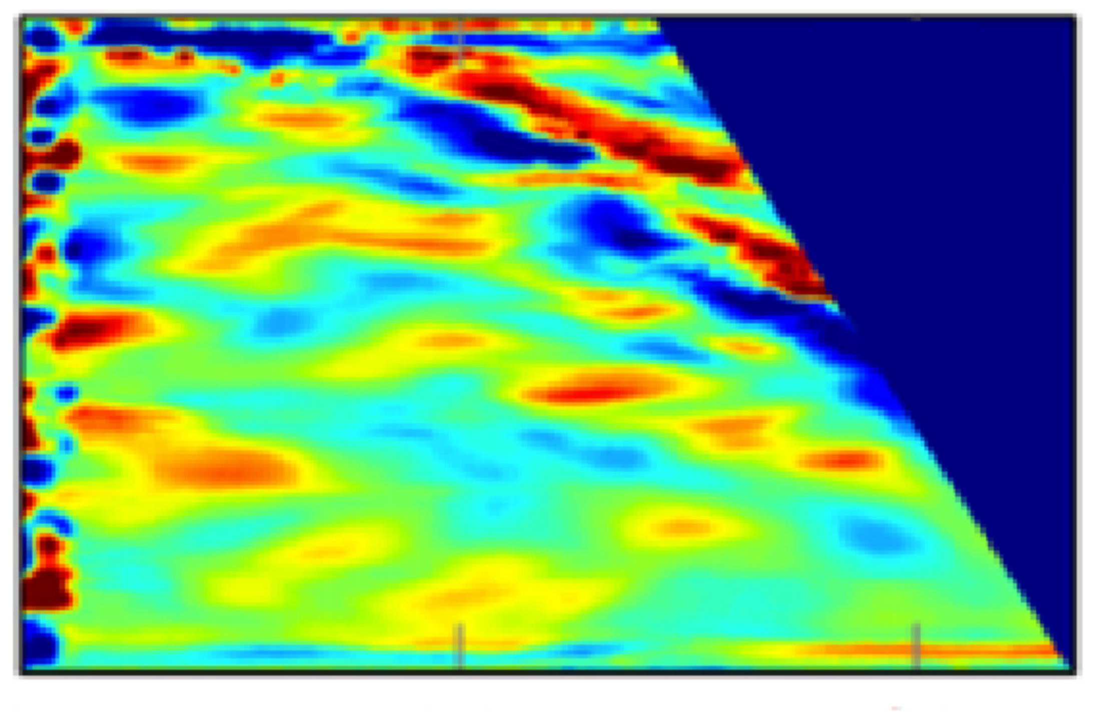


Large forcing





Well developed instability in a wave attractor



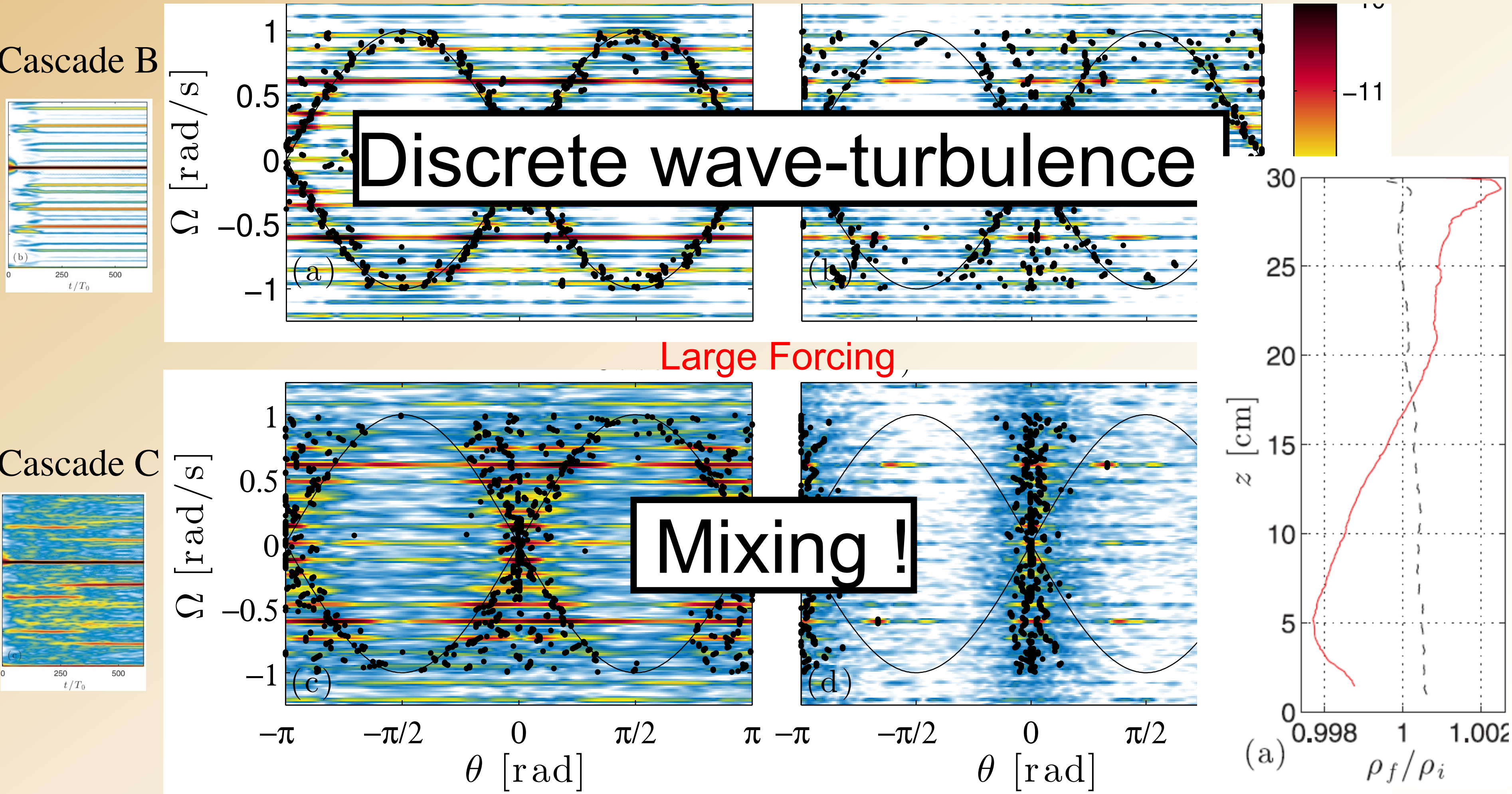
What is beyond internal wave attractor?
Is this wave turbulence?

Wave turbulence?

cf. Yarom & Sharon, Nature Physics 2014

$$\Omega = \pm \sin\theta$$

Intermediate Forcing



Larger wavelengths

Smaller wavelengths

Brouzet, Ermanyuk, Joubaud, Sibgatullin, Dauxois, EPL 113, 44001 (2016)

Mixing

Horizontal vorticity:

$$\xi = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

Gradient Richardson number:

$$Ri = \frac{N^2}{(du/dz)^2}$$

Modified Richardson number:

$$Ri_\xi = \frac{N^2}{\xi^2}$$

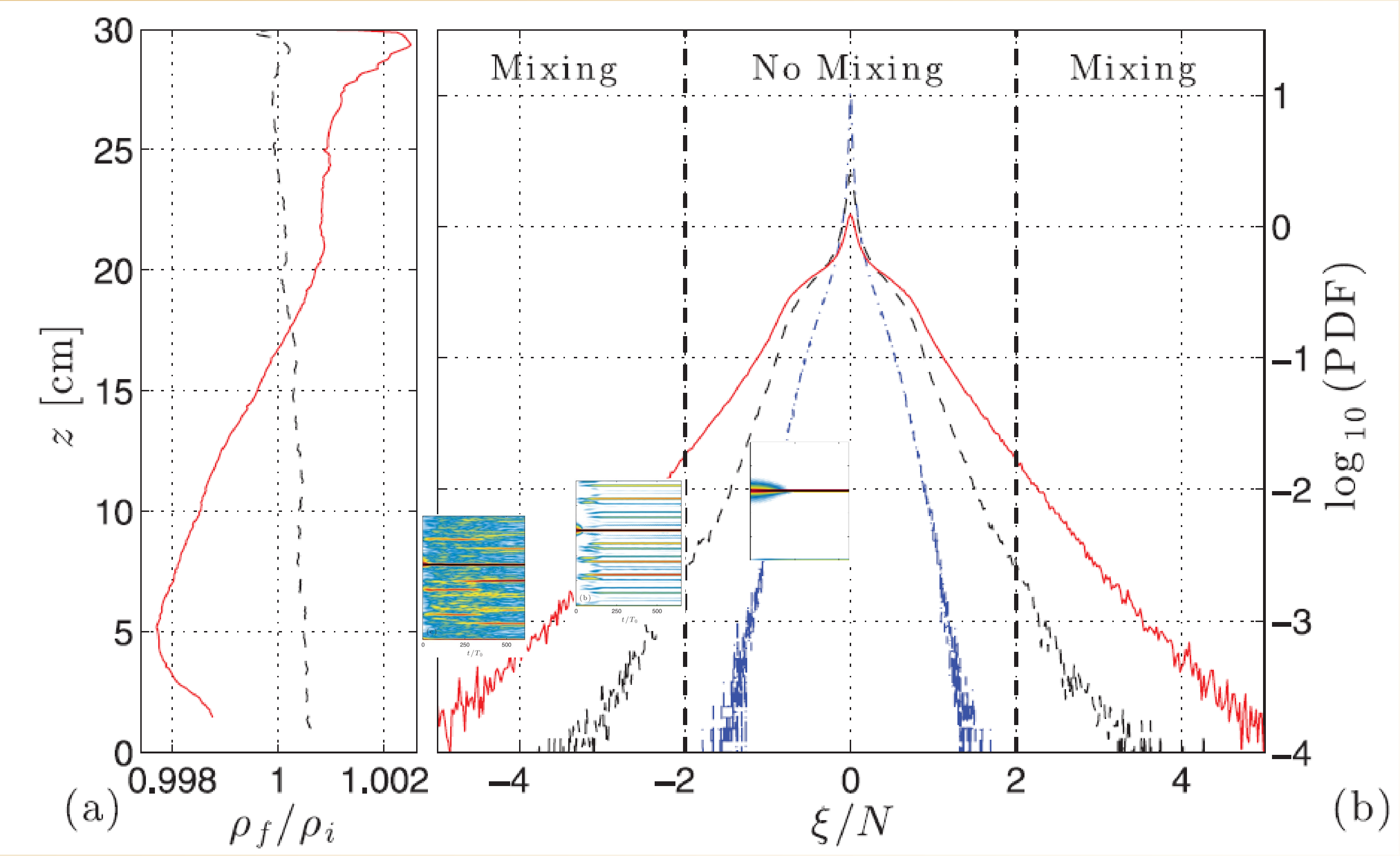
Extension of the Miles-Howard condition

$$Ri > \frac{1}{4}$$

Mixing at

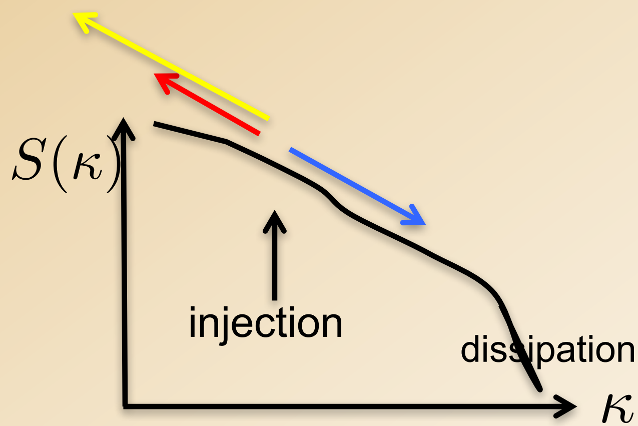
$$\boldsymbol{Ri_\xi < \frac{1}{4}} \longrightarrow \left| \frac{\xi}{N} \right| > 2$$

Statistics of Mixing events



whole-field horizontal vorticity PDF

Fate of Internal Tides



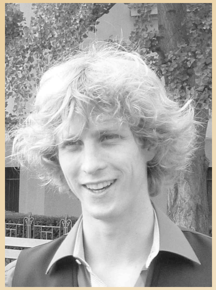
- **<u>Triadic Resonant Instability</u>**: *Direct* and *Inverse* transfers occur simultaneously

- **<u>Streaming instability (mean flow generation):</u>** Inverse transfer
(waterfall rather than a cascade)

- **<u>Internal wave attractor:</u>** Good experimental set-up to study <u>Wave-turbulence</u>
<u>Abyssal mixing</u>

Acknowledgments

C.Brouzet



B.Bourget

G.Bordes

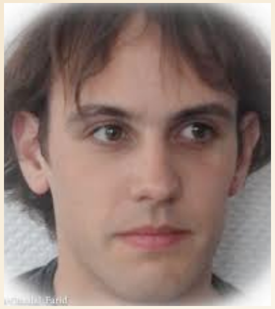
G.Davis

G.Pillet

H.Scolan

S.Joubaud

P.Odier



A. Venaille

E.Ermanyuk

M.LeBars

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Publications

Review	Dauxois, Joubaud, Odier, Venaille, *Annual Review of Fluid Mechanics* (2018)
Triadic Resonant Instability	Bourget, Dauxois, Joubaud, Odier, *JFM* **723**, 1 (2013)
	Bourget, Scolan, Dauxois, Le Bars, Odier, Joubaud *JFM* **759**, 739 (2014)
Mean Flow	Bordes, Venaille, Joubaud, Odier & Dauxois, *PoF* **24**, 086602 (2012)
Attractors: Instabilities and beyond	Scolan, Ermanyuk, Dauxois, *PRL* **110**, 234501 (2013)
	Brouzet, Ermanyuk, Joubaud, Sibgatullin, T. Dauxois, *EPL* **113**, 44001 (2016)
	Brouzet, Sibgatullin, Scolan, Ermanyuk, Dauxois, *JFM* **793**, 109 (2016)

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