Non-canonical Hamiltonian dynamics

Ted Shepherd
Department of Meteorology
University of Reading
• Hamilton’s equations for a canonical system:

\[
\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \quad (i = 1, \ldots, N)
\]

• For a Newtonian potential system, we get Newton’s second law:

\[
\mathcal{H} = (|p|^2/2m) + U(q) \Rightarrow m\frac{d^2q_i}{dt^2} = -\frac{\partial U}{\partial q_i} \quad (i = 1, \ldots, N)
\]

Conservation of energy follows:

(\text{repeated indices summed})

Symplectic formulation:

\[
\frac{du_i}{dt} = J_{ij} \frac{\partial \mathcal{H}}{\partial u_j} \quad (i = 1, \ldots, 2N)
\]

\[
\mathbf{u} = (q_1, \ldots, q_N, p_1, \ldots, p_N)
\]

\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]
• The symplectic formulation of Hamiltonian dynamics can be generalized to other $J$, which have to satisfy certain mathematical properties

• Among these is skew-symmetry, which guarantees energy conservation:

$$\frac{dH}{dt} = \frac{\partial H}{\partial u_i} \frac{du_i}{dt} = \frac{\partial H}{\partial u_i} J_{ij} \frac{\partial H}{\partial u_j} = 0$$

• The canonical $J$ is invertible. If $J$ is non-invertible, then Casimirs are defined to satisfy

$$J_{ij} \frac{\partial C}{\partial u_j} = 0 \quad (i = 1, \ldots, 2N)$$

Casimirs are invariants of the dynamics since

$$\frac{dC}{dt} = \frac{\partial C}{\partial u_i} \frac{du_i}{dt} = \frac{\partial C}{\partial u_i} J_{ij} \frac{\partial H}{\partial u_j} = -\frac{\partial H}{\partial u_i} J_{ij} \frac{\partial C}{\partial u_j} = 0$$
• Example of a non-canonical Hamiltonian representation: Euler’s equations for a rigid body. The dependent variables are the components of angular momentum about principal axes, and the total angular momentum is a Casimir invariant.

• Cyclic coordinates: e.g. rotational symmetry implies conservation of angular momentum

$$\frac{\partial H}{\partial q_i} = 0 \Rightarrow \frac{dp_i}{dt} = 0$$

• More generally, the link between symmetries and conservation laws is provided by Noether’s theorem:

Given a function $F(u)$, define $\delta_F u_i = \varepsilon J_{ij} (\partial F / \partial u_j)$

Then

$$\delta_F H = \frac{\partial H}{\partial u_i} \delta_F u_i = \varepsilon \frac{\partial H}{\partial u_i} J_{ij} \frac{\partial F}{\partial u_j}$$
• But

\[ \frac{d\mathcal{F}}{dt} = \frac{\partial \mathcal{F}}{\partial u_i} \frac{du_i}{dt} = \frac{\partial \mathcal{F}}{\partial u_i} \frac{\partial \mathcal{H}}{\partial u_j} J_{ij} \]

and hence \( \delta_{\mathcal{F}} \mathcal{H} = 0 \) if and only if \( d\mathcal{F}/dt = 0 \)

• Casimir invariants are associated with ‘invisible’ symmetries since

\[ \delta_{\mathcal{C}} u = 0 \]

• Example: rigid body

  – In canonical coordinates, rotational symmetry is explicit and leads to angular momentum conservation through Noether’s theorem

  – In Euler’s equations, angles have been eliminated and the rotational symmetry is now invisible; thus angular momentum (which is still conserved) comes in as a Casimir