Non-canonical Hamiltonian dynamics

Ted Shepherd
Department of Meteorology
University of Reading

- Hamilton's equations for a canonical system:

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad (i = 1, \ldots, \mathrm{N})$$

- For a Newtonian potential system, we get Newton's second law:

$$\mathcal{H} = (|\mathbf{p}|^2/2m) + U(\mathbf{q}) \;\; \Rightarrow \;\; m\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = -\frac{\partial U}{\partial q_i} \qquad (i = 1, \ldots, N)$$

Conservation of energy follows:
(repeated indices summed)

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial \mathcal{H}}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}$$

$$= \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} = 0$$

Symplectic formulation:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = J_{ij}\frac{\partial \mathcal{H}}{\partial u_j} \qquad (i = 1, \ldots, 2N)$$

$$\mathbf{u} = (q_1, \ldots, q_{\mathrm{N}}, p_1, \ldots, p_{\mathrm{N}})$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- The symplectic formulation of Hamiltonian dynamics can be generalized to other *J*, which have to satisfy certain mathematical properties
- Among these is skew-symmetry, which guarantees energy conservation:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial u_i}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial u_i} J_{ij} \frac{\partial \mathcal{H}}{\partial u_j} = 0$$

- The canonical J is invertible. If J is non-invertible, then Casimirs are defined to satisfy

$$J_{ij}\frac{\partial C}{\partial u_j} = 0 \qquad (i = 1, \ldots, 2N)$$

Casimirs are invariants of the dynamics since

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\partial C}{\partial u_i}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial C}{\partial u_i} J_{ij} \frac{\partial \mathcal{H}}{\partial u_j} = -\frac{\partial \mathcal{H}}{\partial u_i} J_{ij} \frac{\partial C}{\partial u_j} = 0$$

- Example of a non-canonical Hamiltonian representation: Euler's equations for a rigid body. The dependent variables are the components of angular momentum about principal axes, and the total angular momentum is a Casimir invariant.
- Cyclic coordinates: e.g. rotational symmetry implies conservation of angular momentum

$$\frac{\partial H}{\partial q_i} = 0 \Rightarrow \frac{dp_i}{dt} = 0 \qquad \text{for a given } i$$

- More generally, the link between symmetries and conservation laws is provided by *Noether's theorem*:

Given a function $\mathcal{F}(\mathbf{u})$, define $\delta_{\mathcal{F}} u_i = \varepsilon J_{ij}(\partial \mathcal{F}/\partial u_j)$

Then $$\delta_{\mathcal{F}}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial u_i}\delta_{\mathcal{F}} u_i = \varepsilon \frac{\partial \mathcal{H}}{\partial u_i} J_{ij} \frac{\partial \mathcal{F}}{\partial u_j}$$

- But $$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \frac{\partial \mathcal{F}}{\partial u_i}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \mathcal{F}}{\partial u_i} J_{ij} \frac{\partial \mathcal{H}}{\partial u_j}$$

  and hence $\delta_{\mathcal{F}}\mathcal{H} = 0$ if and only if $\mathrm{d}\mathcal{F}/\mathrm{d}t = 0$

- Casimir invariants are associated with 'invisible' symmetries since

$$\delta_C \mathbf{u} = 0$$

- Example: rigid body

  – In canonical coordinates, rotational symmetry is explicit and leads to angular momentum conservation through Noether's theorem

  – In Euler's equations, angles have been eliminated and the rotational symmetry is now invisible; thus angular momentum (which is still conserved) comes in as a Casimir