Pseudomomentum and wave, mean-flow interaction

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On the 'wave momentum' myth

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'The question is,' said Alice, 'whether you *can* make words mean different things.'

'The question is,' said Humpty Dumpty, 'which is to be master – that's all.'

– Carroll (1871)

Barotropic beta-plane: $f = f_0 + \beta y$ (y represents latitude)

$$q = \nabla^2\psi + \beta y\,, \qquad \frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi + \beta y) = 0$$

(N.B. Need to have $L \ll f_0/\beta$)

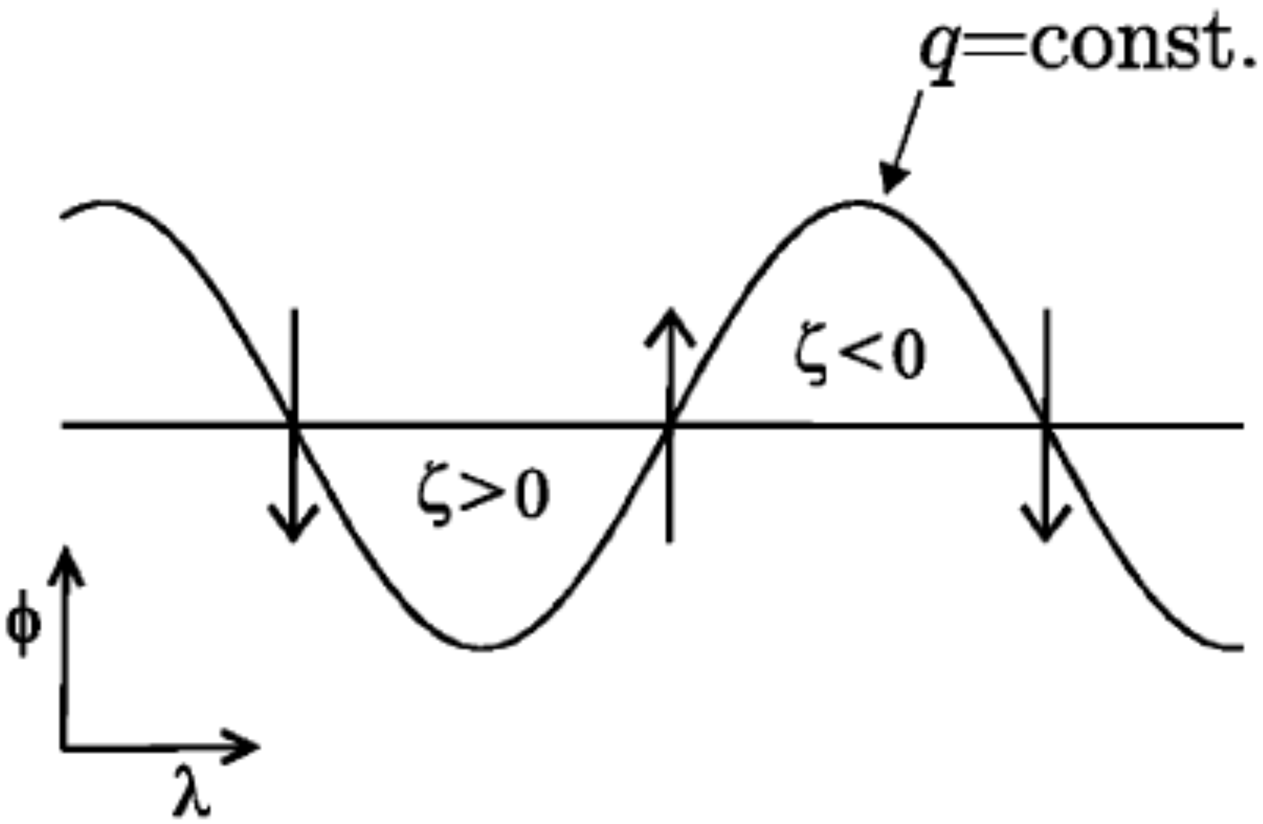
- Linearize about a constant zonal flow U:

$$\frac{\partial}{\partial t}\nabla^2\psi + U\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0$$

- The ansatz $\exp\{i(kx + \ell y - \omega t)\}$ leads to $\omega = Uk - \dfrac{\beta k}{k^2 + \ell^2}$
- These are **Rossby waves** (note unidirectional!)
  - Account for stationary and low-frequency flow structures in the atmosphere and ocean
  - Also are the building blocks of synoptic-scale eddies (macroturbulence), though are not linear waves in this case

- **Schematic of Rossby-wave propagation**

  An undulation in the PV contour (with constant $\zeta + \beta y$, where ζ is vorticity) induces $\zeta < 0$ north of the rest position and $\zeta > 0$ south of the rest position, which act constructively in inducing a velocity field that moves the undulation left (to the west)



- **Barotropic dynamics** is a Hamiltonian system

$$\frac{\partial \omega}{\partial t} = -\mathbf{v} \cdot \nabla \omega = -\partial(\psi, \omega)$$

$$\begin{aligned}
\delta \mathcal{H} &= \delta \iint \frac{1}{2} |\nabla \psi|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \iint \nabla \psi \cdot \delta \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \\
&= \iint \{\nabla \cdot (\psi \delta \nabla \psi) - \psi \delta \omega\} \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

$\longrightarrow \delta \mathcal{H} / \delta \omega = -\psi$

(assuming boundary terms vanish)

- Functional derivatives are just the infinite-dimensional analogue of partial derivatives; they can reflect non-local properties

- Barotropic dynamics can be written in symplectic form as:

$$\frac{\partial \omega}{\partial t} = J\frac{\delta \mathcal{H}}{\delta \omega} \quad \text{where} \quad J = -\partial(\omega, \cdot) \qquad \delta\mathcal{H}/\delta\omega = -\psi$$

- The Casimir invariants are:

$$\mathcal{C} = \iint \mathrm{C}(\omega)\,\mathrm{d}x\,\mathrm{d}y \quad \text{with} \quad \frac{\delta \mathcal{C}}{\delta \omega} = \mathrm{C}'(\omega)$$

  and correspond to Lagrangian conservation of vorticity

- Symmetry in *x* and conservation of *x*-momentum:

$$-\varepsilon\frac{\partial \omega}{\partial x} = \delta_{\mathcal{M}}\omega = \varepsilon J\frac{\delta \mathcal{M}}{\delta \omega} = -\varepsilon\partial\left(\omega, \frac{\delta \mathcal{M}}{\delta \omega}\right)$$

$$\delta\mathcal{M}/\delta\omega = y. \longrightarrow \mathcal{M} = \iint y\omega\,\mathrm{d}x\,\mathrm{d}y = \iint y\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Kelvin's impulse $= \iint u\,\mathrm{d}x\,\mathrm{d}y$ (ignoring boundary terms)

- **Disturbance invariants:** arguably the most powerful application of Hamiltonian geophysical fluid dynamics
- Ambiguities about the energy of a wave…
- Ambiguities about the momentum of a wave…
- If *u*=*U* is a steady solution of a Hamiltonian system, then

$$J \frac{\delta \mathcal{H}}{\delta \mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{U}} = 0$$

- For a canonical system, *J* is invertible so $\delta\mathcal{H}/\delta\mathbf{u} = 0 \text{ at } \mathbf{u} = \mathbf{U}.$
  - Hence the disturbance energy is quadratic
- But for a non-canonical system, this is not true and the disturbance energy is generally linear in the disturbance
  - Not sign-definite
  - Cannot define stability, normal modes, etc.
  - Leads to concept of *pseudoenergy*

- **Pseudomomentum:** In a similar manner, if a basic state $u=U$ is independent of x (i.e. is invariant with respect to translation in x), then by Noether's theorem,

$$\partial\mathbf{U}/\partial x = 0 \quad \text{implies} \quad J\frac{\delta\mathcal{M}}{\delta\mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{U}} = 0$$

which implies $\delta(\mathcal{M}+\mathcal{C}) = 0$ at $\mathbf{u}=\mathbf{U}$ for some Casimir C

$\mathcal{A} = (\mathcal{M}+\mathcal{C})[\mathbf{u}] - (\mathcal{M}+\mathcal{C})[\mathbf{U}]$ is then both conserved and quadratic in the disturbance

(*pseudomomentum*)

- Example: **Barotropic flow on the beta-plane**

$$\frac{\delta\mathcal{M}}{\delta q} = y, \qquad \frac{\delta\mathcal{C}}{\delta q} = C'(q)$$

- Consider disturbances to an x-invariant basic state $q_0(y)$

$$\delta(\mathcal{M}+\mathcal{C}) = 0 \text{ at } q = q_0 \quad \text{implies} \quad C'(q_0) = -y$$

- This is analogous to the formula for APE, and similarly,

$$\mathcal{A} = \iint \left\{ -\int_0^{q-q_0} [Y(q_0+\tilde{q}) - Y(q_0)]\, \mathrm{d}\tilde{q} \right\} \mathrm{d}x\, \mathrm{d}y$$

  where $Y(q_0(y)) = y$; which is negative definite for $\mathrm{d}q_0/\mathrm{d}y > 0$

- Small-amplitude approximation: $-\dfrac{(q-q_0)^2}{2(\mathrm{d}q_0/\mathrm{d}y)}$

- If q_0 is defined to be the zonal mean, then $q_0 = \overline{q}, \quad q' = q - \overline{q}$

  and the zonal mean of this expression becomes $-\dfrac{\overline{q'^2}}{2\bar{q}_y}$

- Exactly the same form applies to stratified QG flow, where the negative of this quantity is known as the **Eliassen-Palm (E-P) wave activity**

- N.B. The sign of this quantity corresponds to the sign of the intrinsic frequency of Rossby waves (negative if $\mathrm{d}q_0/\mathrm{d}y > 0$)

- **Relationship between pseudomomentum and momentum:** consider the zonally averaged zonal momentum equation for the barotropic beta-plane:

$$\frac{\partial \bar{u}}{\partial t} = -\overline{\frac{\partial u^2}{\partial x}} - \overline{\frac{\partial uv}{\partial y}} + f\bar{v} - \overline{\frac{\partial p}{\partial x}} = -\frac{\overline{\partial u'v'}}{\partial y}$$

$$\frac{\partial \bar{u}}{\partial t} = -\overline{v'\left(\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}\right)} + \overline{\frac{\partial}{\partial x}\left[\tfrac{1}{2}(u'^2 - v'^2)\right]} = \overline{v'q'}$$

- The linearized potential-vorticity equation is

$$\frac{\partial q'}{\partial t} + \bar{u}\frac{\partial q'}{\partial x} + v'\frac{\mathrm{d}\bar{q}}{\mathrm{d}y} = 0$$

and hence (if $\bar{q}_y \neq 0$) $\quad v' = -\frac{1}{\bar{q}_y}\left(\frac{\partial q'}{\partial t} + \bar{u}\frac{\partial q'}{\partial x}\right)$

$$\longrightarrow \quad \overline{q'v'} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\overline{q'^2}}{\bar{q}_y}\right) = \frac{\partial \bar{A}}{\partial t} \quad \text{whence} \quad \frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{A}}{\partial t}$$

(Taylor identity)

- **Stratified QG dynamics:** zonal-wind tendency equation, temperature tendency equation, and thermal-wind balance together imply

$$\mathcal{L}\Big(\frac{\partial \bar{u}}{\partial t}\Big) = \frac{\partial^2}{\partial y^2}\overline{(v'q')} \quad \text{where} \quad \mathcal{L} = \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{\rho_0}{S}\frac{\partial}{\partial z}$$

- So it's the same physics, but the zonal-wind response to mixing of potential vorticity is now *spatially non-local* (the Eliassen balanced response): follows from PV inversion
- The pseudomomentum conservation law takes the local form (with *S* being a source/sink)

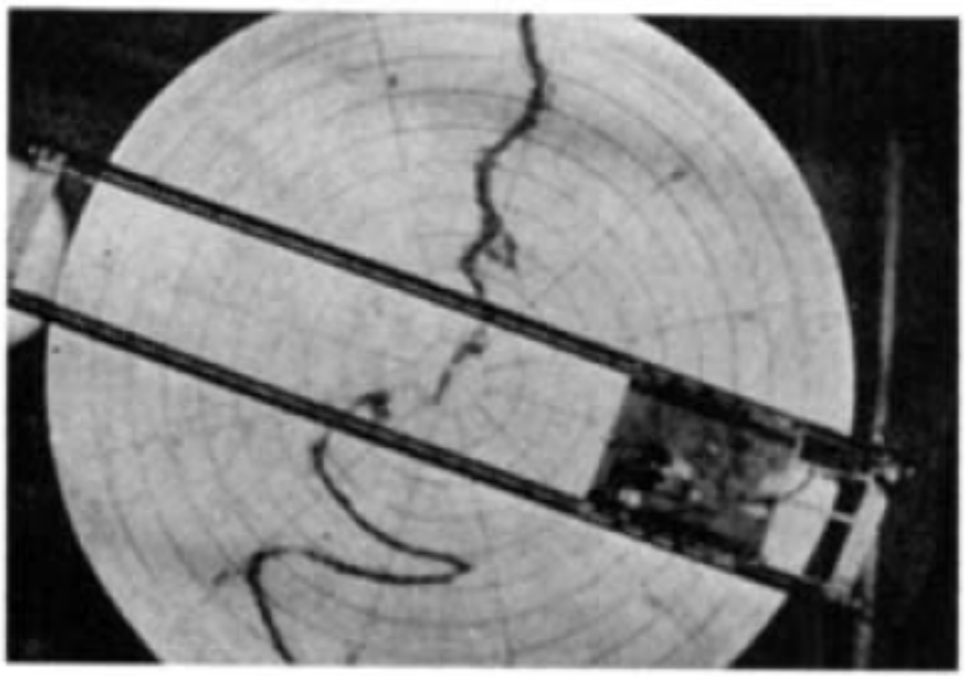
$$\frac{\partial A}{\partial t} + \nabla \cdot \boldsymbol{F} = S \qquad\qquad \nabla \cdot \boldsymbol{F} = -\overline{v'q'}$$

$$\mathcal{L}\Big(\frac{\partial \bar{u}}{\partial t}\Big) = \frac{\partial^2}{\partial y^2}\overline{(v'q')} = -\frac{\partial^2}{\partial y^2}\nabla\cdot\bar{\boldsymbol{F}} = \frac{\partial^2}{\partial y^2}\Big(\frac{\partial \bar{A}}{\partial t} - \bar{S}\Big)$$

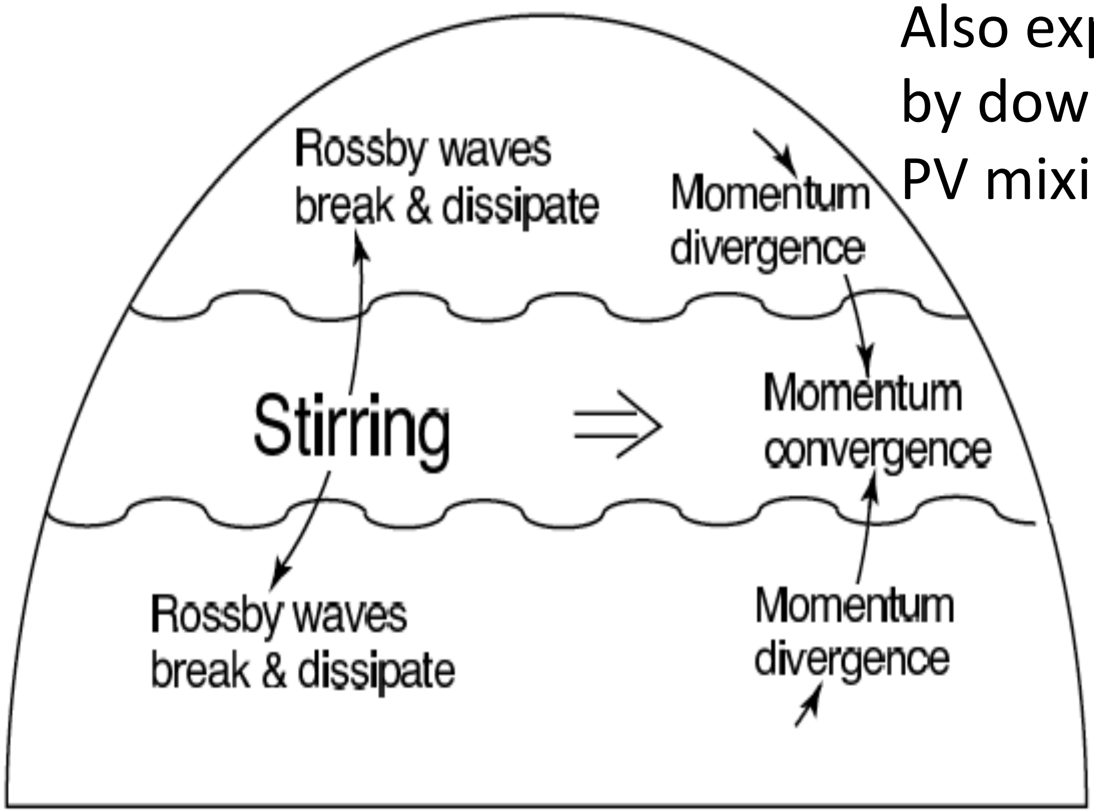
- So mean-flow changes require wave transience or non-conservative effects (*non-acceleration theorem*)

- In the atmosphere, we can generally assume that $\bar{q}_y > 0$ since q is dominated by β
- Hence $A < 0$; *Rossby waves carry negative pseudomomentum*
- Where Rossby waves dissipate, there must be a convergence of negative pseudomomentum, hence a negative torque
- Conservation of momentum implies a positive torque in the wave source region
- This phenomenon is seen in laboratory rotating-tank experiments
- A prograde jet emerges from random stirring, surrounded on either side by retrograde jets (seen in distortion of dye)

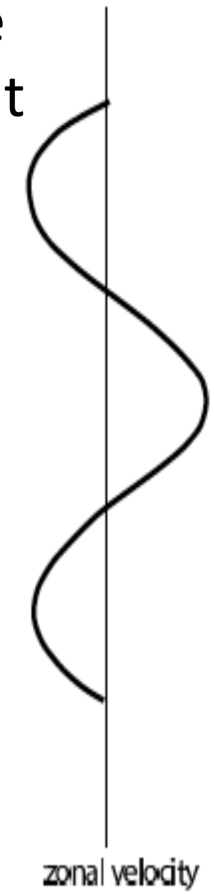
(Whitehead 1975 Tellus)



- In the atmosphere, synoptic-scale Rossby waves are generated by baroclinic instability, hence within a jet region
- Flux of negative pseudomomentum out of jet corresponds to an upgradient flux of momentum into the jet: "eddy-driven jet"
- For Rossby waves, $sgn(c_{gy}) = sgn(-\overline{u'v'})$



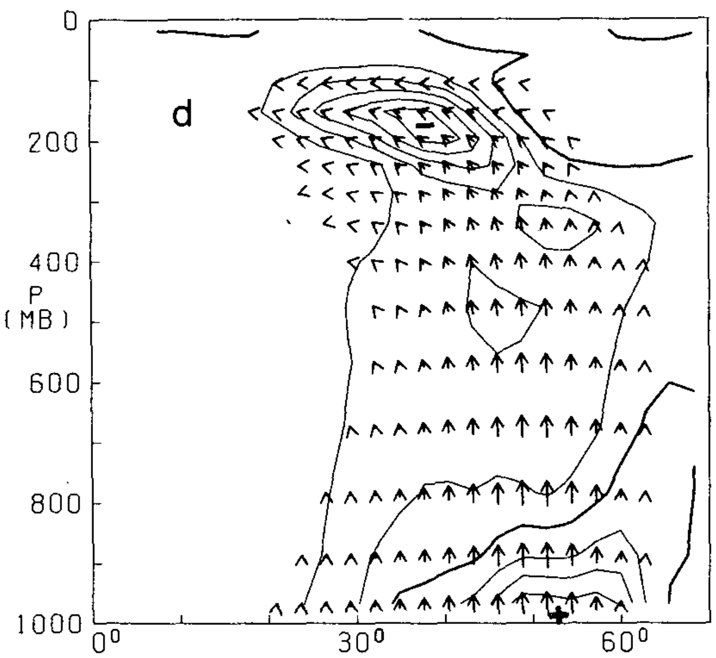
Also explainable by downgradient PV mixing



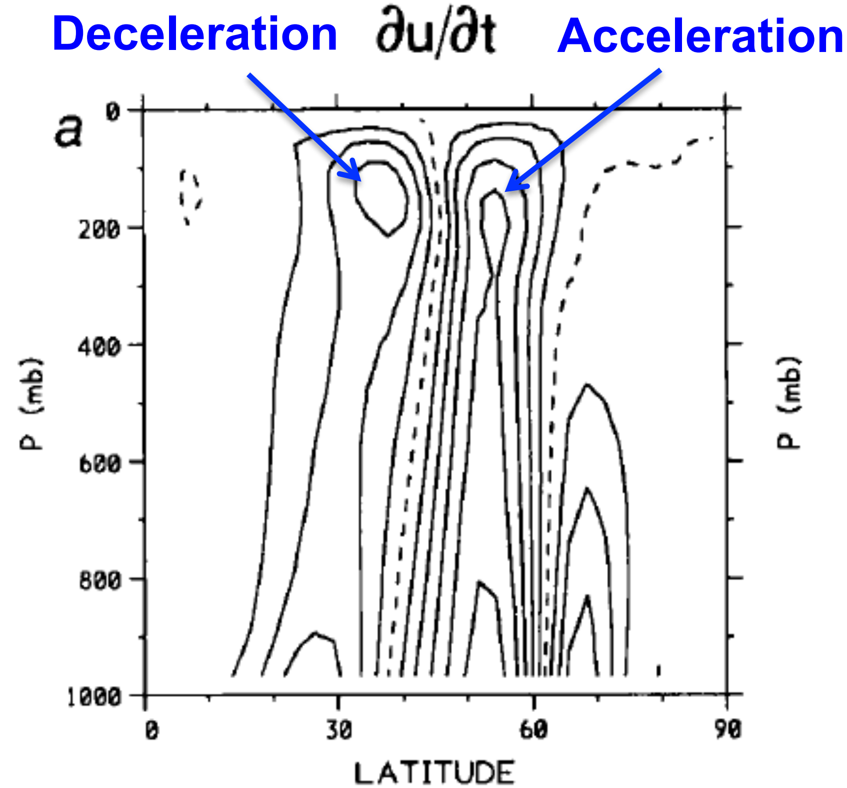
Vallis (2006)

- In fact the wave propagation is up and out (generally equatorward), as seen in these 'baroclinic life cycles' showing baroclinic growth and barotropic decay (Simmons & Hoskins 1978 JAS)

E-P flux (arrows) and divergence (contours)

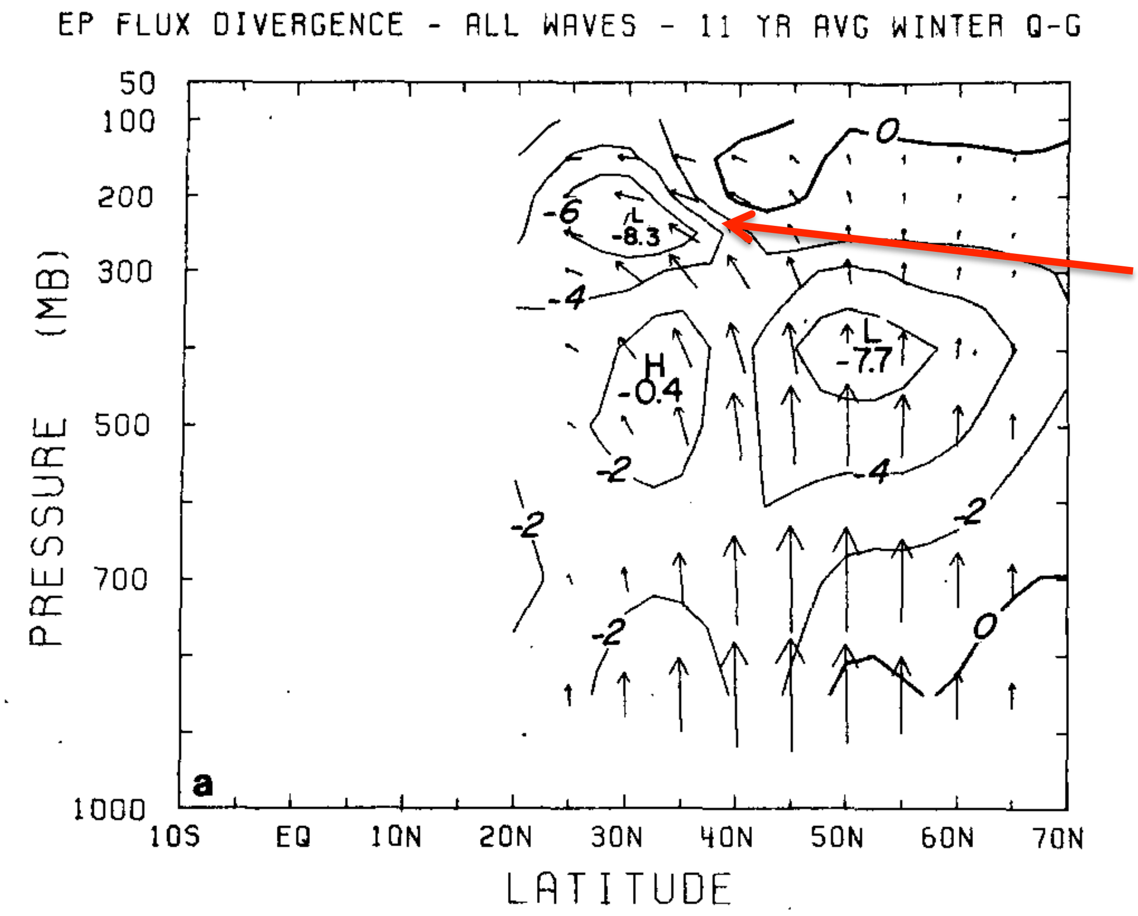


Edmon, Hoskins & McIntyre (1980 JAS)



Haynes & Shepherd (1989 QJRMS)

- The first analysis of EP flux divergence using atmospheric data showed two regions of convergence in the upper troposphere

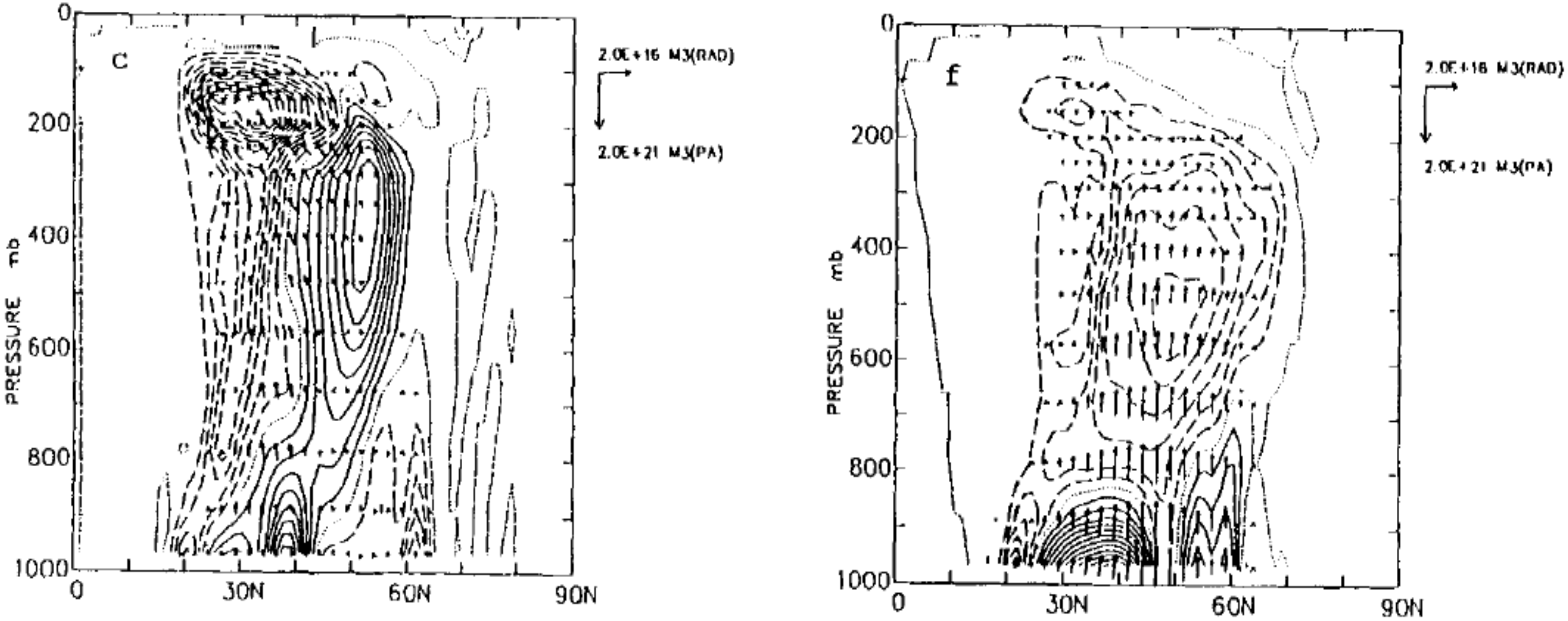


The one in the subtropical upper troposphere resembled that in the baroclinic life cycle of Simmons & Hoskins (1978 JAS)

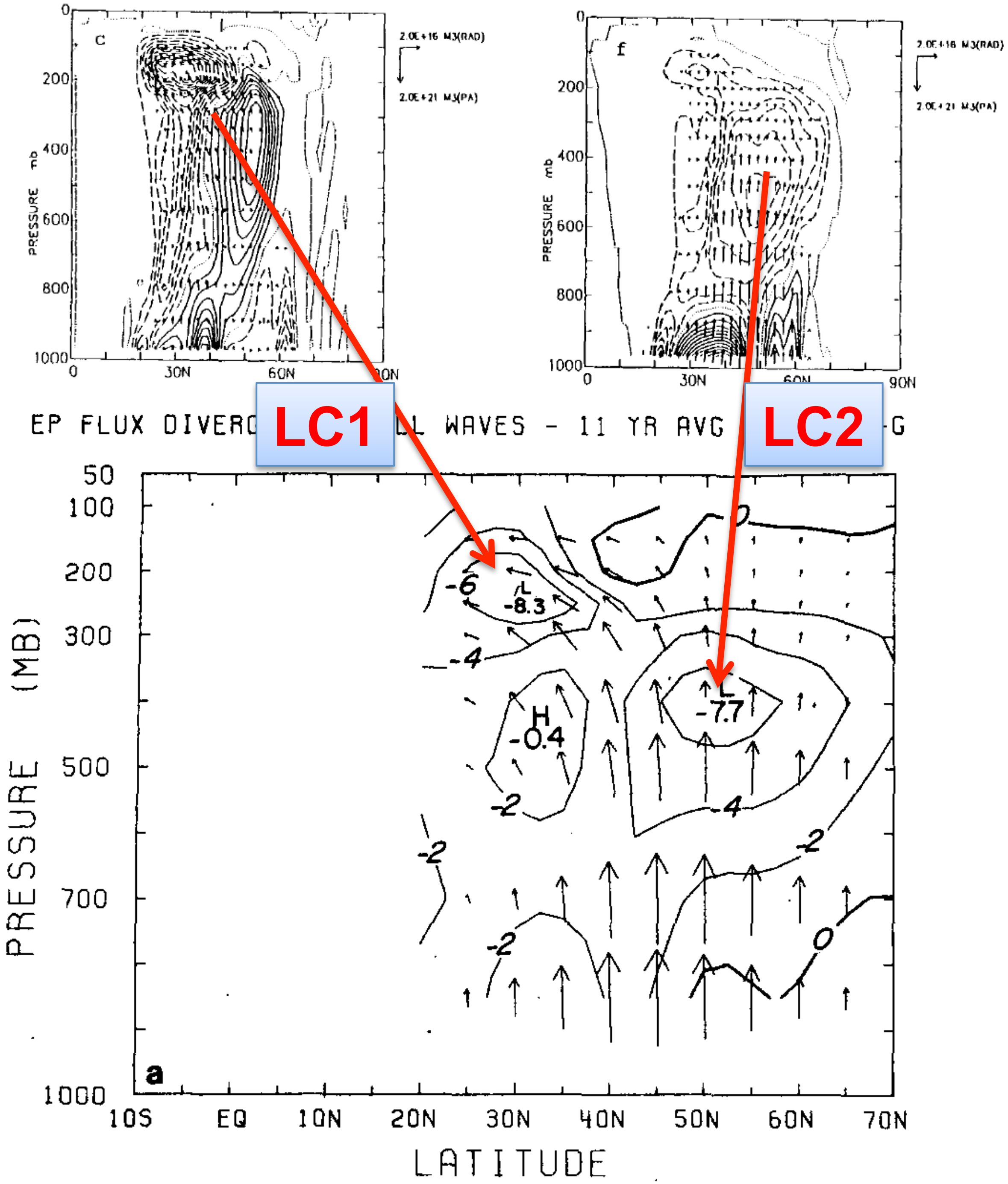
Edmon, Hoskins & McIntyre (1980 JAS)

- Small changes in the upper tropospheric zonal winds caused the baroclinic life cycle to decay instead in the midlatitude middle troposphere, reproducing the other observed feature

“LC1” **EP fluxes at Day 8** **“LC2”**

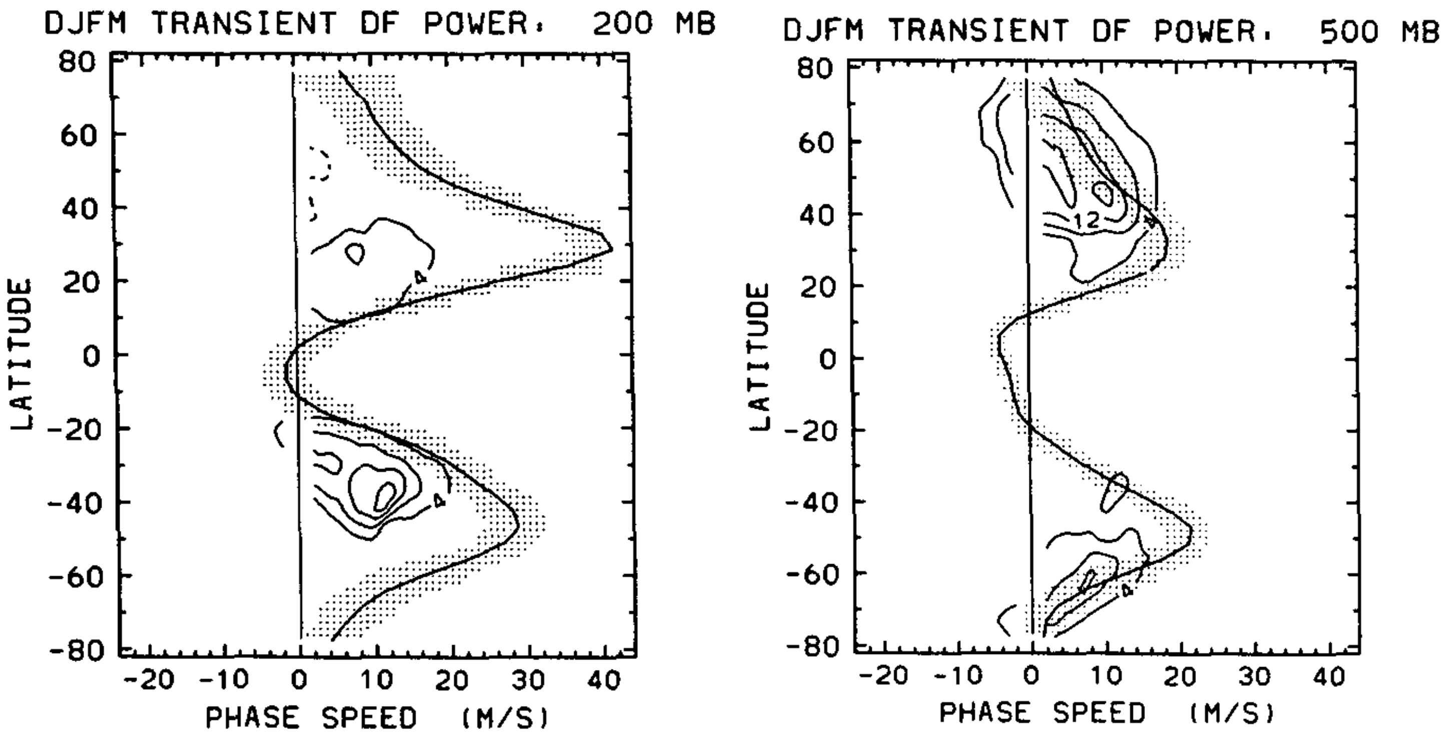


Thorncroft, Hoskins & McIntyre (1993 QJRMS)



The implication is that the atmosphere exhibits both regimes from time to time

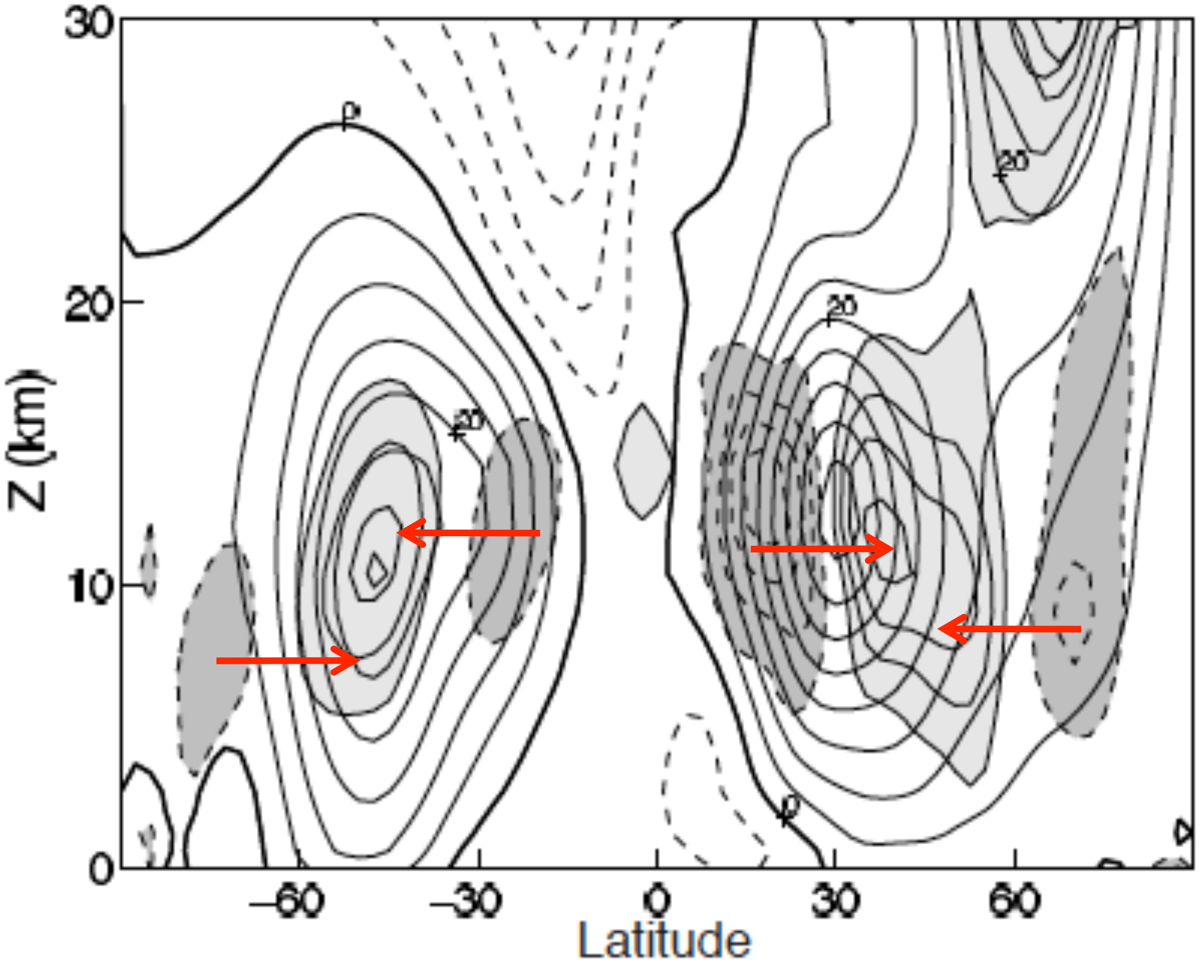
- The observed breaking of synoptic-scale waves occurs in nonlinear critical layers
  - The subtropical critical layer in the upper troposphere, and the midlatitude critical layer in the middle troposphere
  - Here for northern winter; northern summer is similar
  - **Hence the jet both shapes, and is shaped by, the eddies**



McIntyre: **"wave-turbulence jigsaw puzzle"**

Randel & Held (1991 JAS)

- Horizontal eddy momentum fluxes are directed into the jet cores, i.e. upgradient, so **the eddies act to maintain the jet**



Shown for NH winter / SH summer (DJF)

Shading is the eddy horizontal momentum flux convergence

Contours are zonal wind

Vallis (2006)

Perfect alignment in SH where jet is eddy driven

Not perfect alignment in NH where jet involves Hadley circulation