2D and shallow-water turbulence

Ted Shepherd
Department of Meteorology
University of Reading

Barotropic vorticity equation (2D Euler flow)

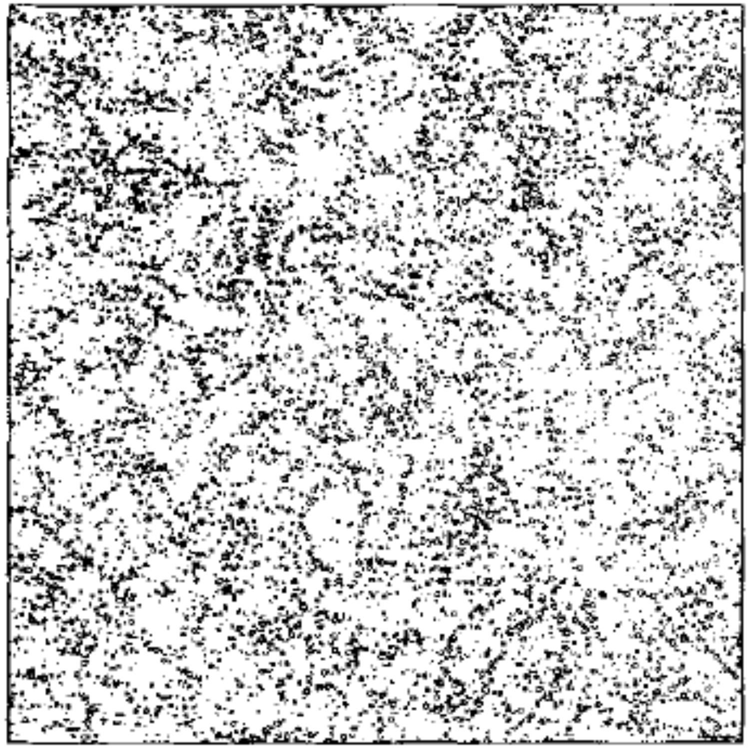
$$q = \nabla^2\psi = \hat{z}\cdot\nabla\times\vec{u} \quad \text{(vorticity)}, \quad \frac{\partial}{\partial t}\nabla^2\psi + J(\psi,\nabla^2\psi) = 0$$

- Has two inviscid integral invariants, the (kinetic) energy and the enstrophy

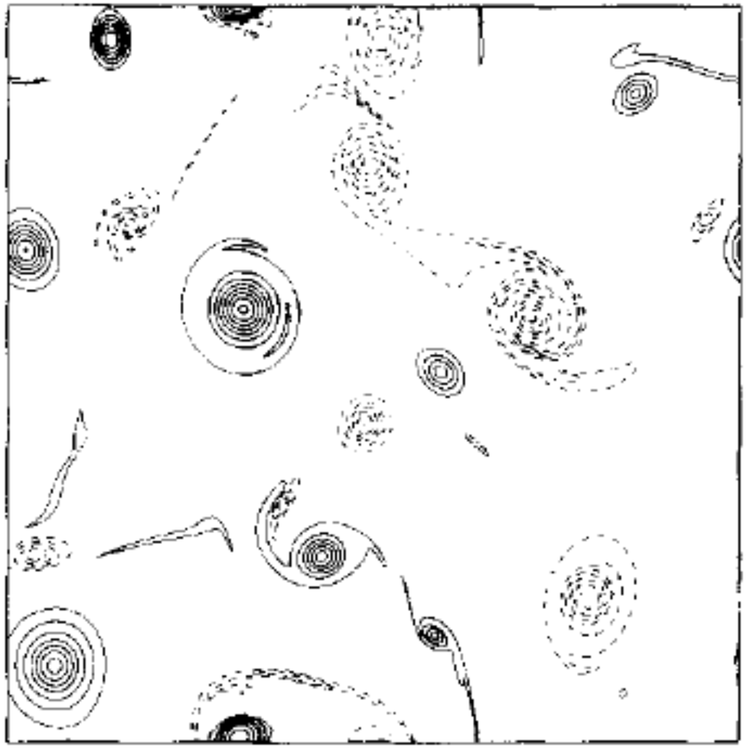
$$\iint \tfrac{1}{2}\left|\nabla\psi\right|^2 \, dxdy \qquad\qquad \iint \tfrac{1}{2}(\nabla^2\psi)^2 \, dxdy$$

- The wavenumber spectra are related by $Z(k)=k^2E(k)$
- Enstrophy conservation prohibits a direct (downscale) energy cascade (a characteristic of 3-D turbulence), and leads to the peculiar properties of **2-D turbulence** (e.g. inverse cascade)
  - Accounts for the large-scale flow structures of geophysical fluids, and the formation of coherent vortices
  - Also a classical problem in mathematics and physics!

- **Spontaneous emergence of coherent vortices**



$t = 0$



$t = 21$

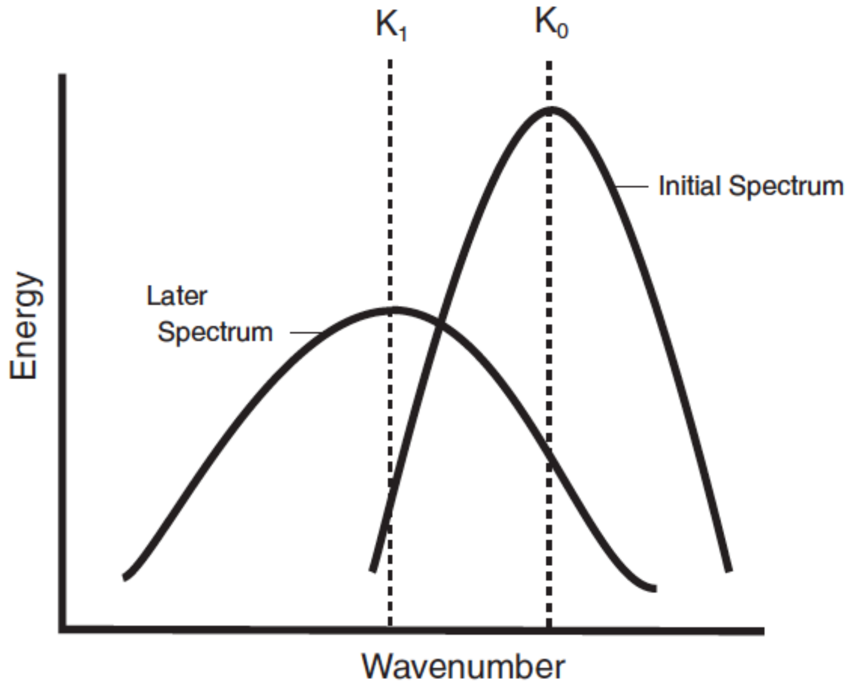
Bartello & Warn (1996 *J.Fluid Mech*.)

- Spreading of an *initially localized* energy spectrum

$$\frac{d}{dt}\int (k-k_0)^2 E(k)\,dk = \frac{d}{dt}\int \left[k^2 E(k) - 2k\,k_0 E(k) + k_0^2 E(k)\right] dk > 0$$

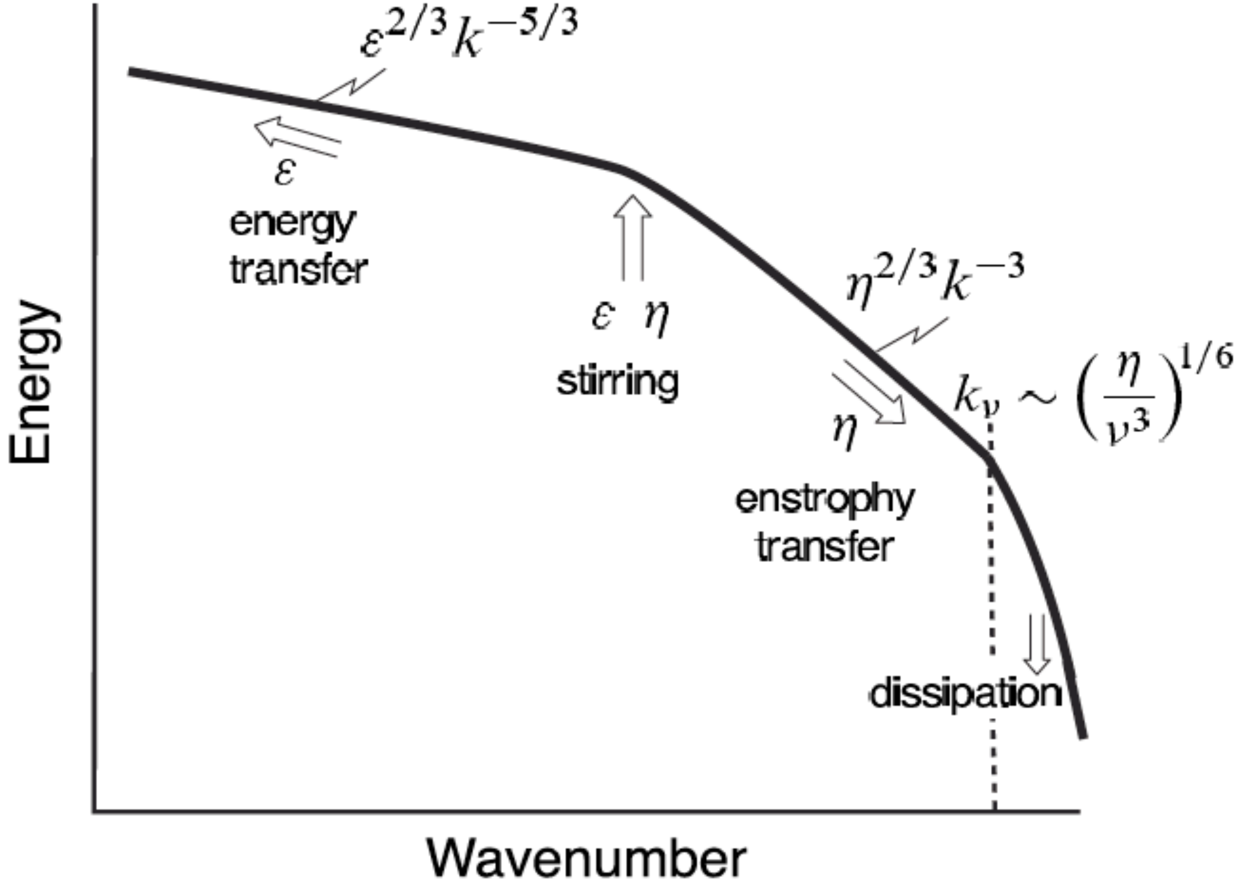
$$\Rightarrow \quad \frac{d}{dt}\int k\,E(k)\,dk < 0$$

- Hence energy moves mainly to smaller *k*, i.e. to larger spatial scales
- Similarly, enstrophy is expected to move mainly to larger *k*



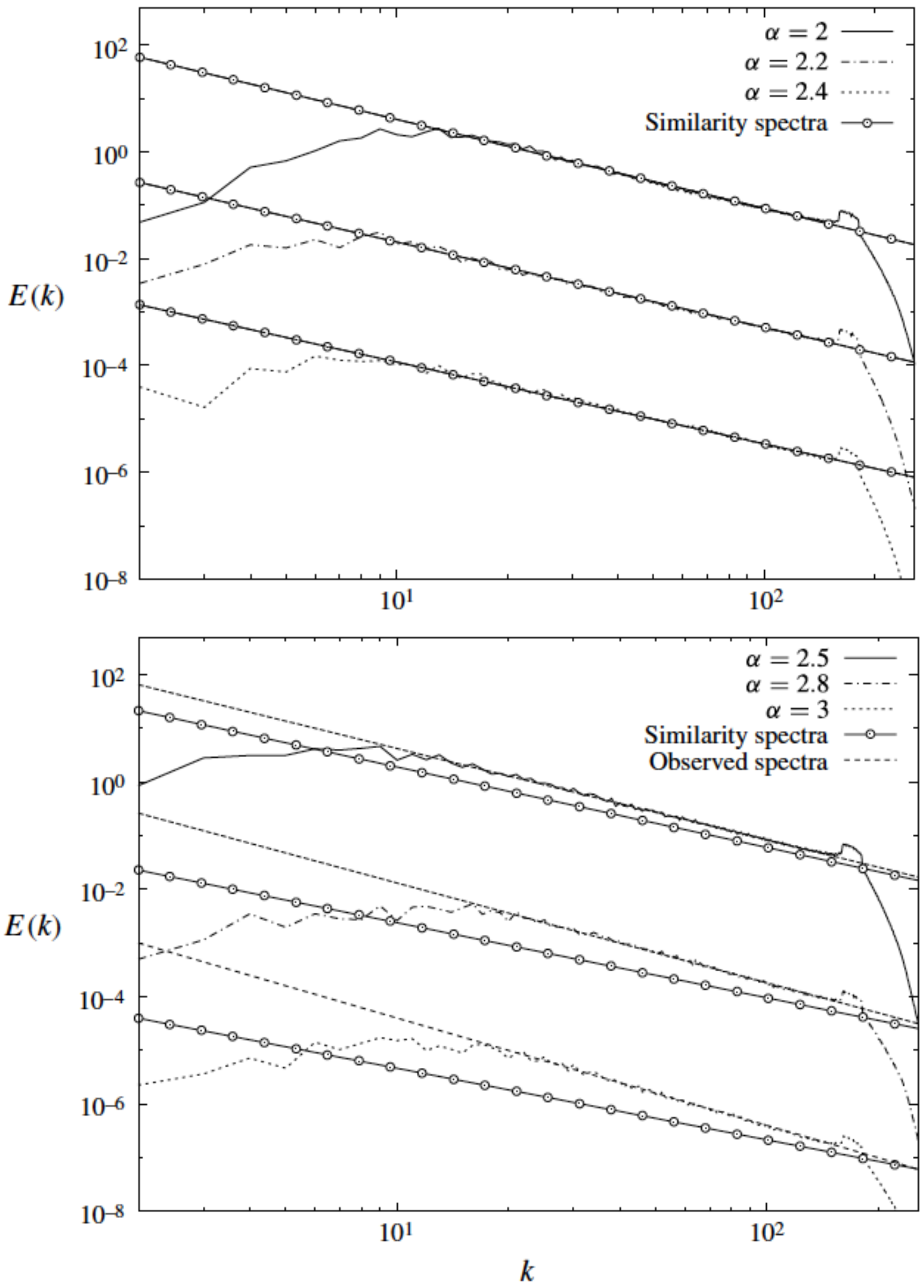
Vallis (2006)

- The classical picture of **two-dimensional turbulence** (after Kraichnan 1967 Phys. Fluids)
  - Power laws follow from scaling symmetry (dim'l analysis)
  - Argued to be relevant to the atmosphere by Charney (1971 JAS)



Energy and (potential) enstrophy injection is by **baroclinic instability**

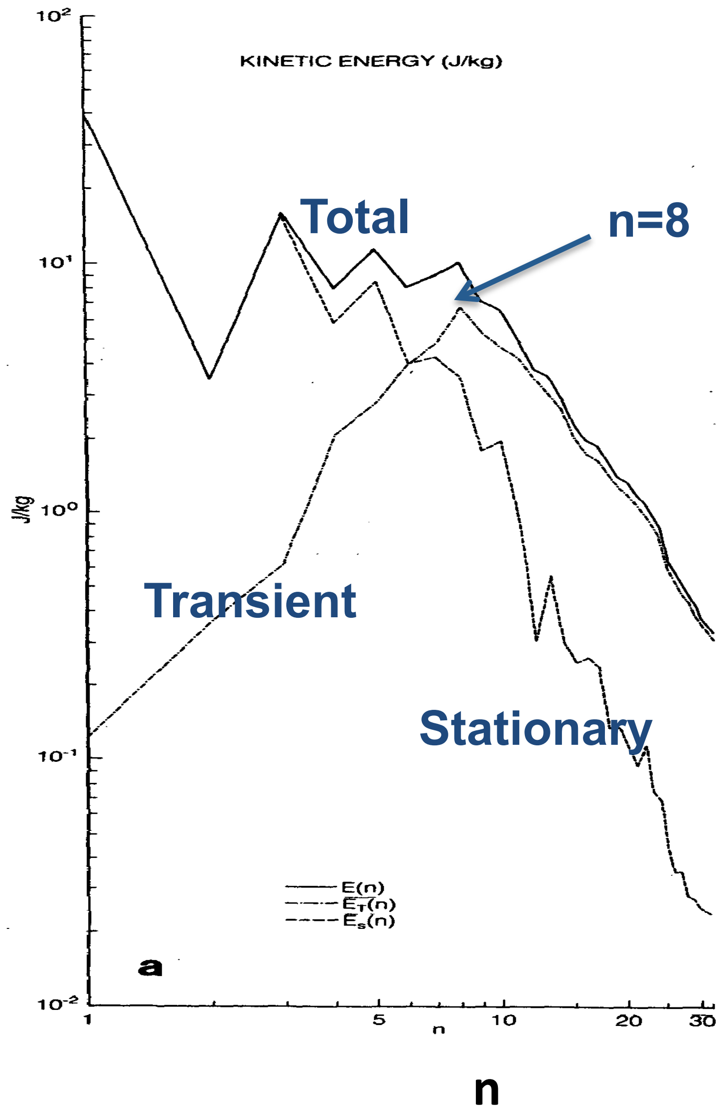
Vallis (2006)

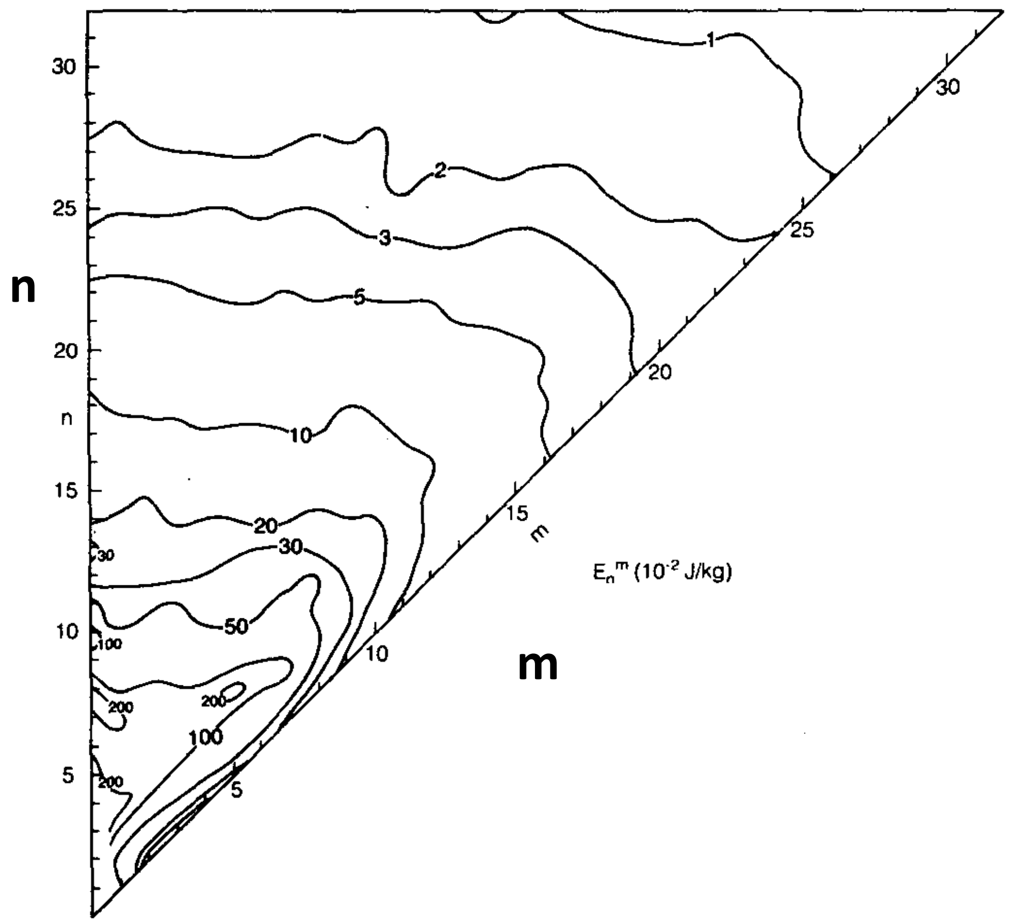


- **An aside: the inverse energy cascade in alpha-turbulence**
- System forced within dissipation range to suppress coherent vortex formation
- No large-scale dissipation; simulation stopped before turbulence is "boxed in"
- Similarity spectrum holds for α < 2.5, but not for α ≥ 2.5!

Burgess & Shepherd (2013 JFM)

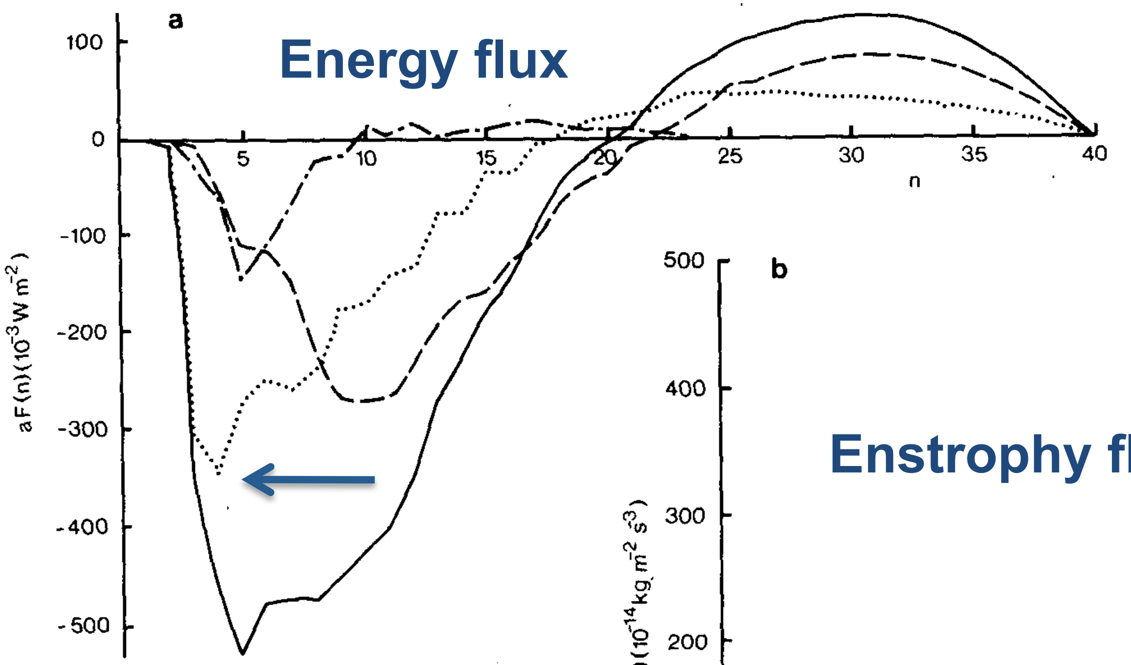
Kinetic energy spectra from FGGE data



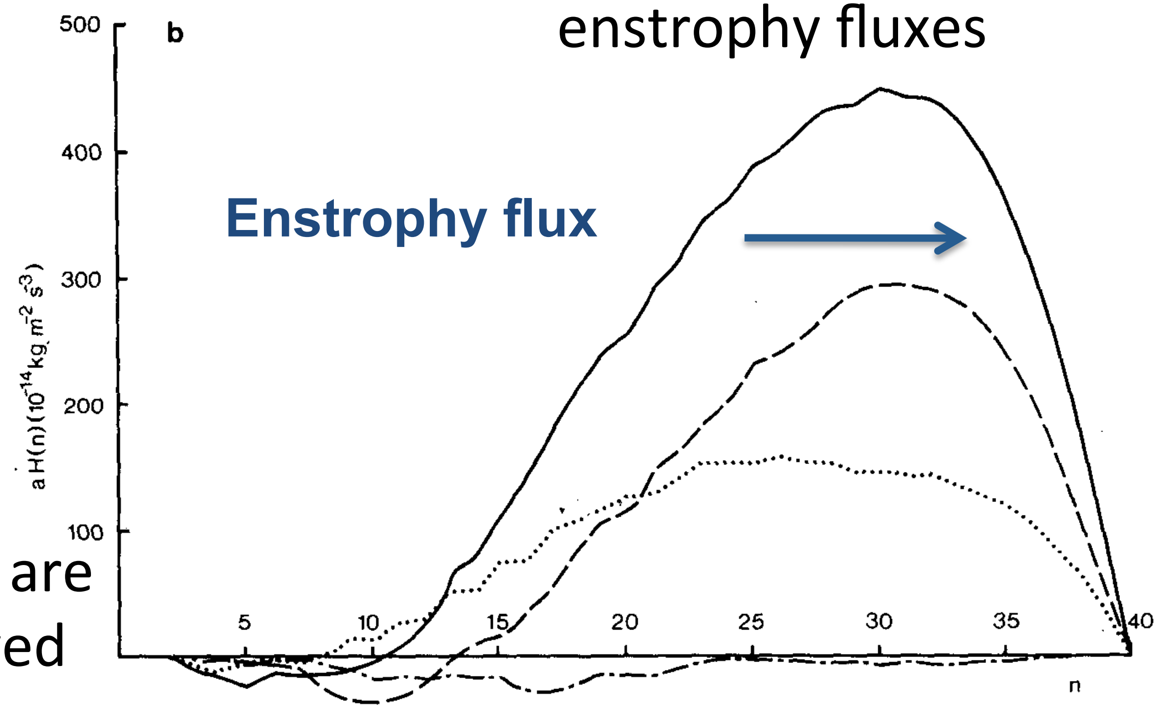


Boer & Shepherd (1983 JAS)

- The spectral fluxes can be decomposed into stationary (dash-dot), transient (dashed), and mixed stationary-transient (dotted) components



The mixed component is a very significant part of both the energy and the enstrophy fluxes

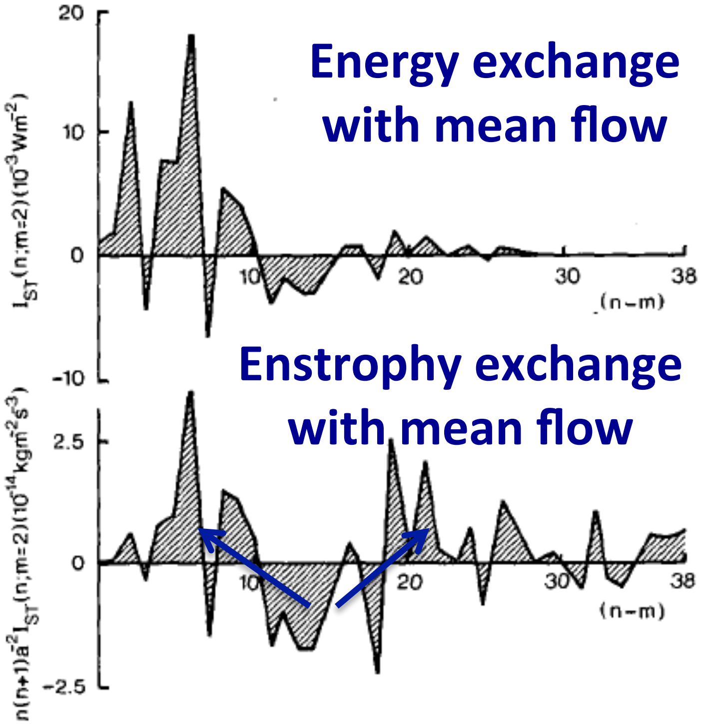


However these fluxes are clearly not well resolved by this data

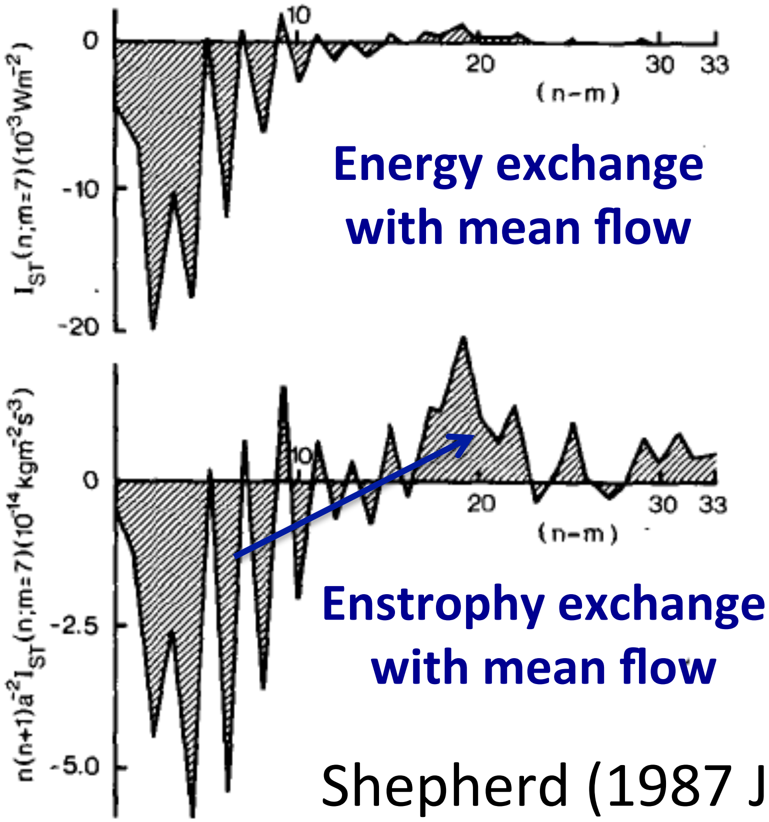
Shepherd (1987 JAS)

- The interaction between a zonal flow and eddies induces non-linear transfer of eddy enstrophy at fixed zonal wavenumber *m*
- Has implications for energy exchange between eddies and mean flow

m=2: eddies gain energy from mean flow

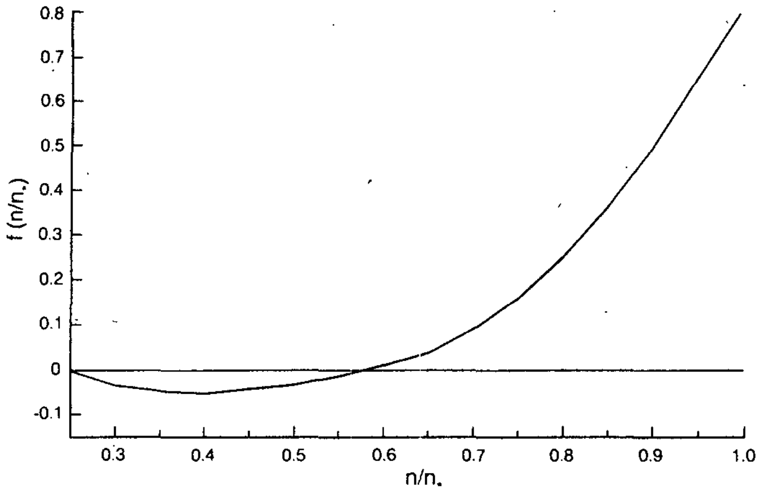


m=7: eddies lose energy to mean flow

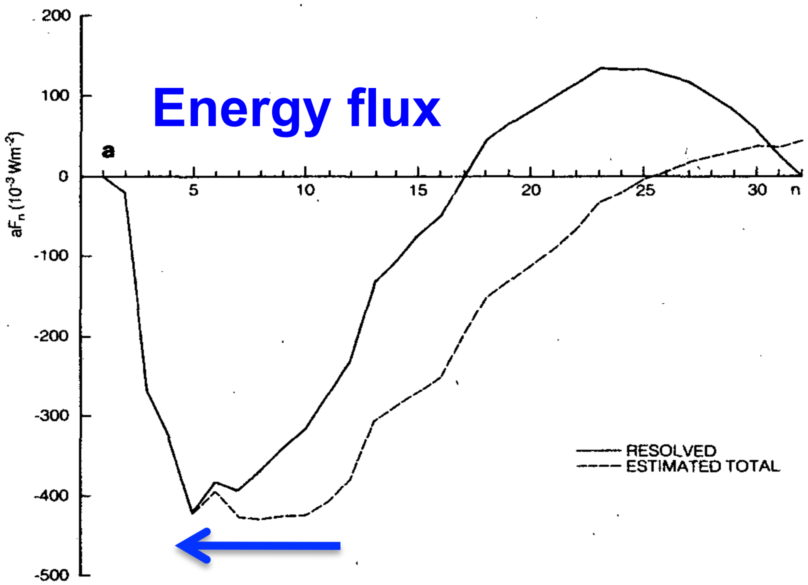


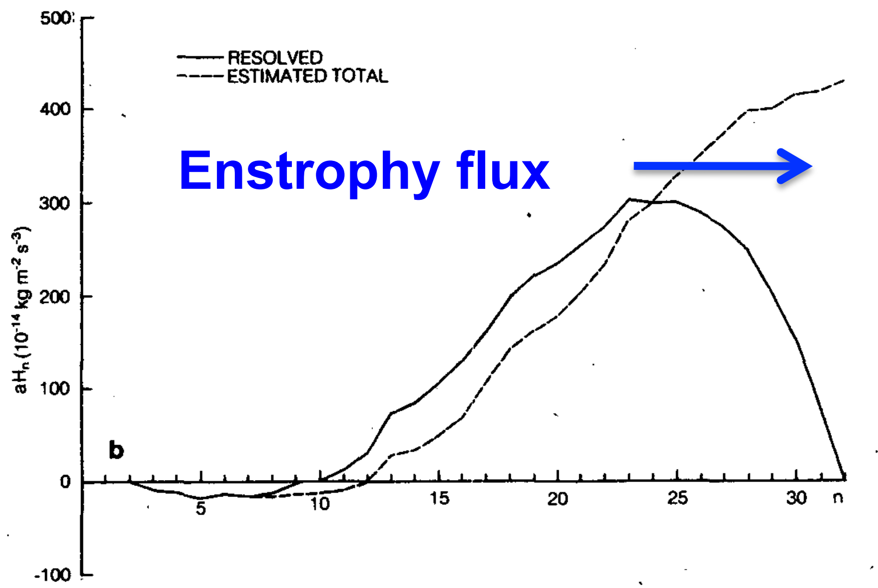
Shepherd (1987 JAS)

- Assuming 2-D turbulence, Leith (1971 JAS) represented the interactions with unresolved scales as an effective diffusion with a negative spectral range, giving zero energy loss (right)

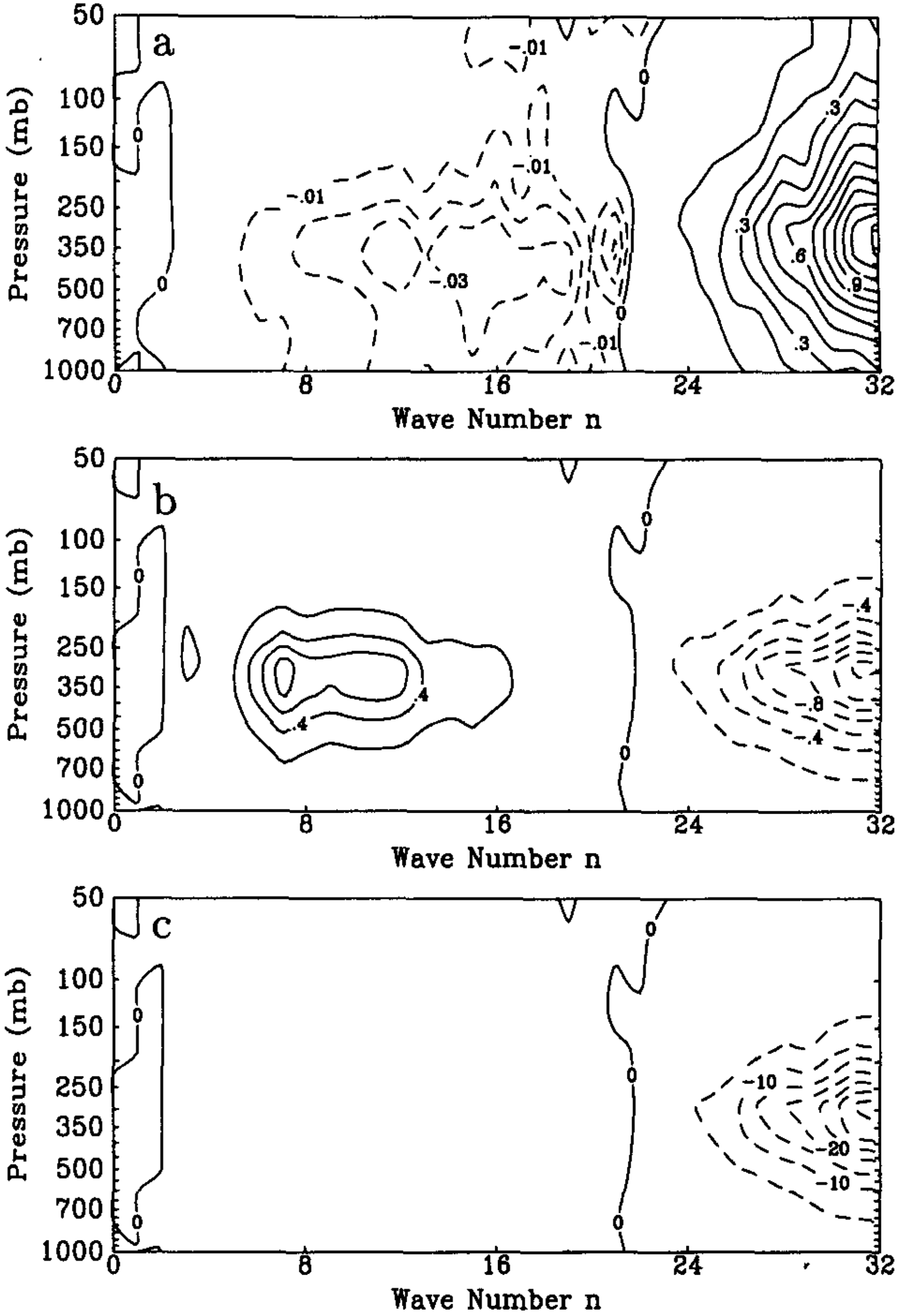


- Applying this to the FGGE data gave estimated total energy and enstrophy fluxes which were consistent with theory





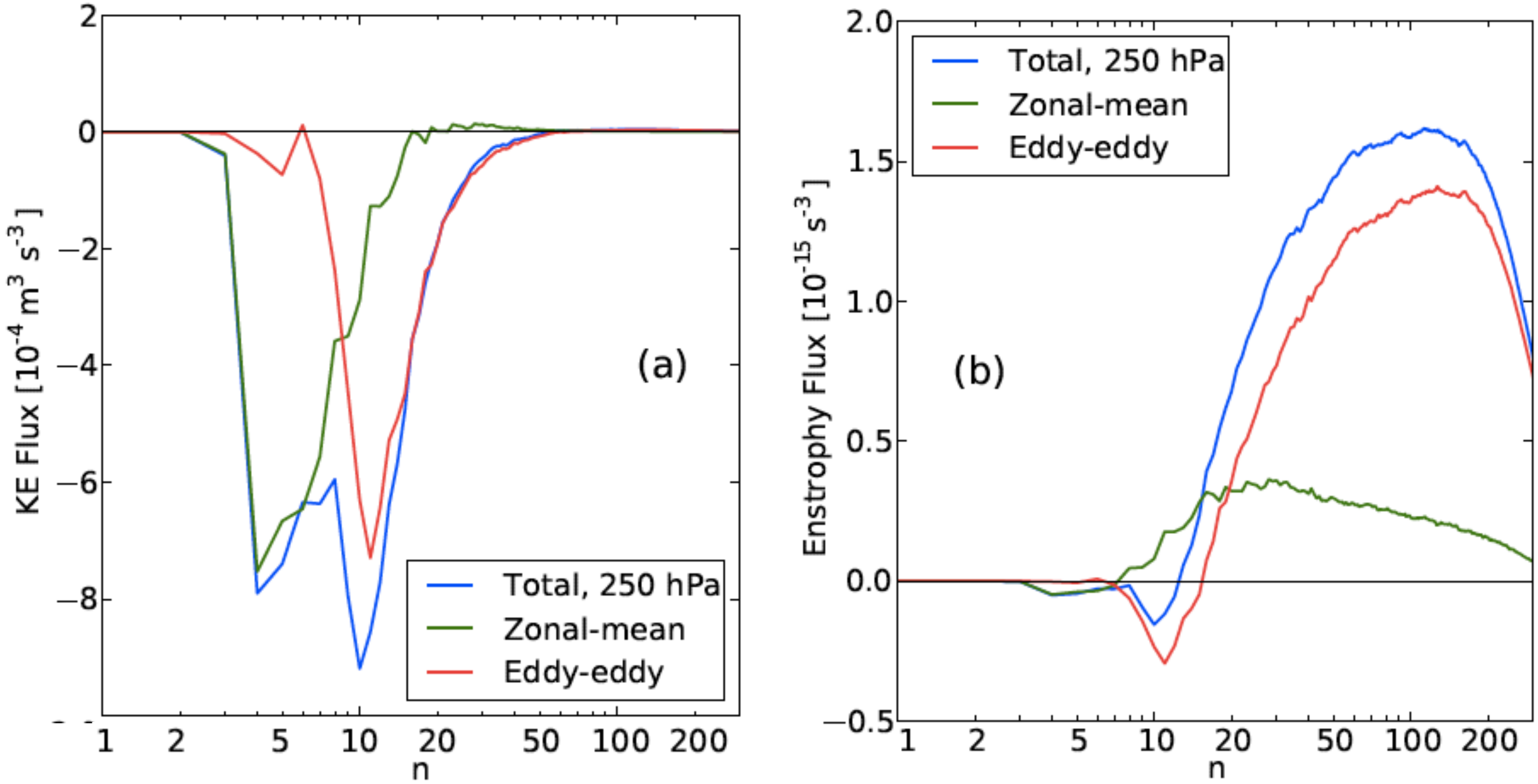
Boer & Shepherd (1983 JAS)



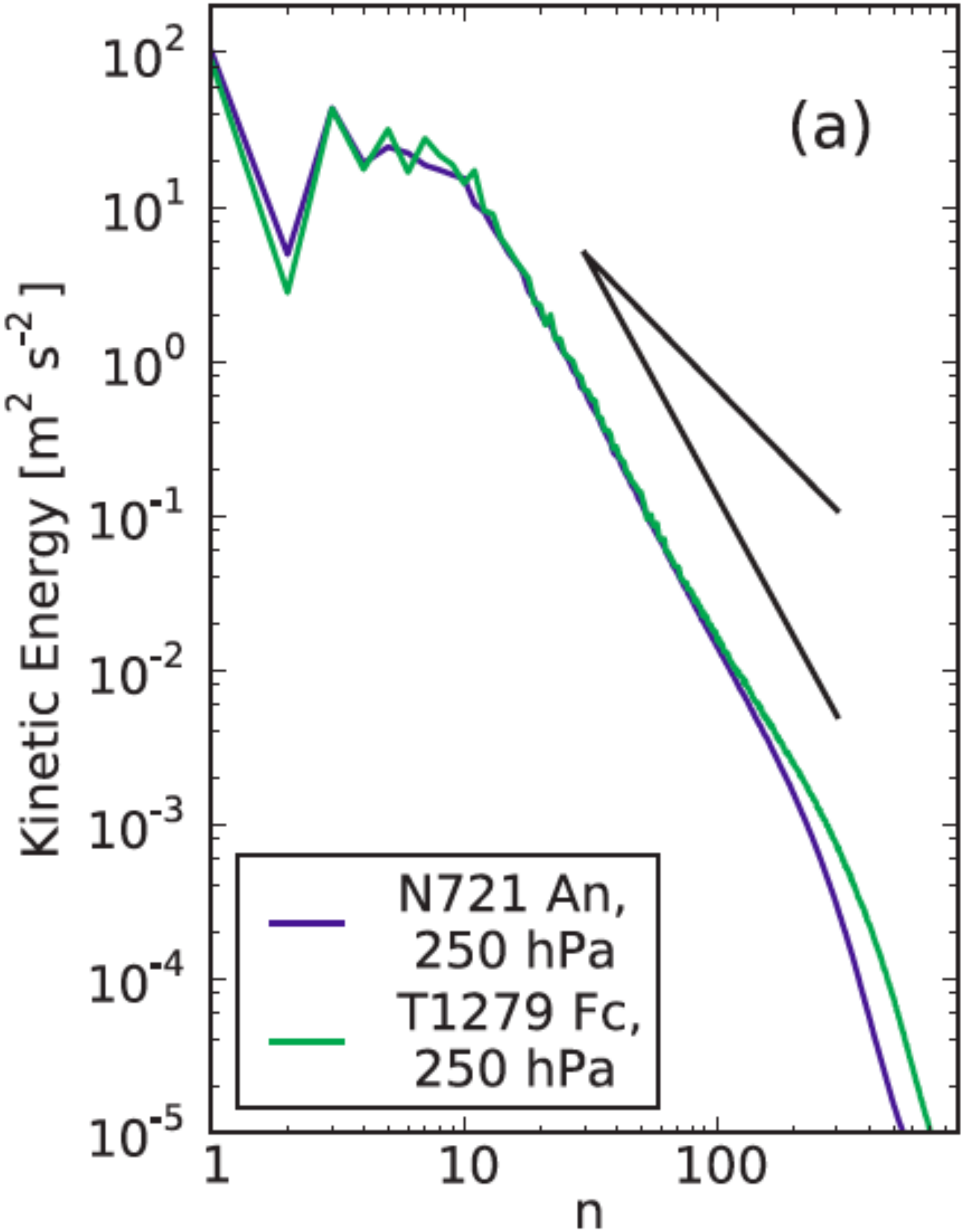
- Using higher-resolution analyses (here ECMWF truncated to T60), the "Leith function" can be estimated for n=0-32 (top panel; note factor of 10 difference in positive and negative C.I.'s)
- Lower panels show the corresponding energy and enstrophy interactions with scales smaller than n=32 (maximizing in upper troposphere)
- Note the energy "backscatter"

Koshyk & Boer (1995 JAS)

- The T799 ECMWF operational analysis from January 2008 appears to resolve the fluxes in the upper troposphere
  - Baroclinic excitation occurs over n=10-30
  - Well defined downscale enstrophy flux, mainly eddy-eddy



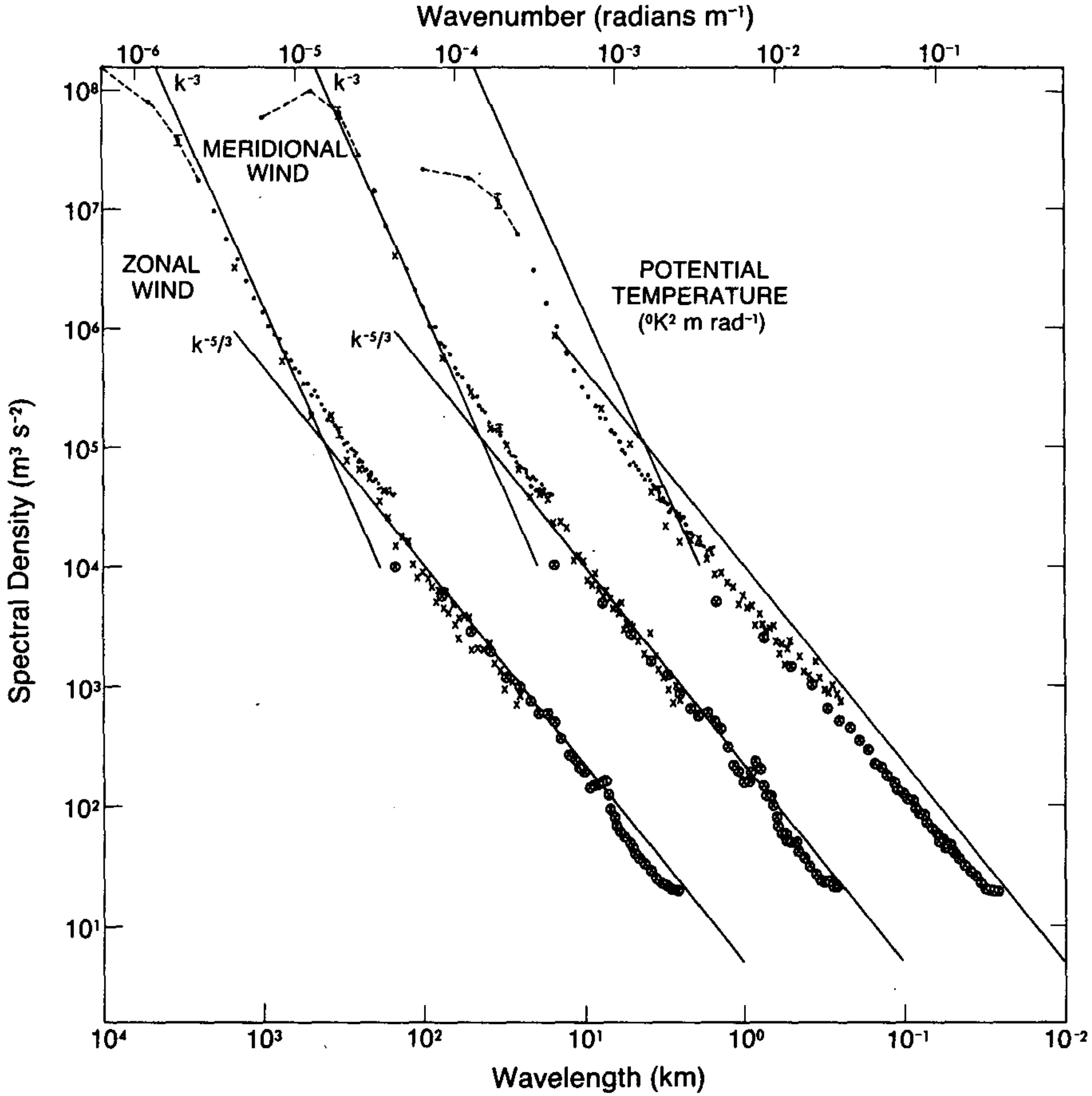
Burgess, Erler & Shepherd (2013 JAS)



- Moreover it gives a remarkably clean k^{-3} energy spectrum in the upper troposphere!

Burgess, Erler & Shepherd (2013 JAS)

- However, upper tropospheric aircraft observations revealed a **$k^{-5/3}$ energy spectrum** at scales from about 5-500 km

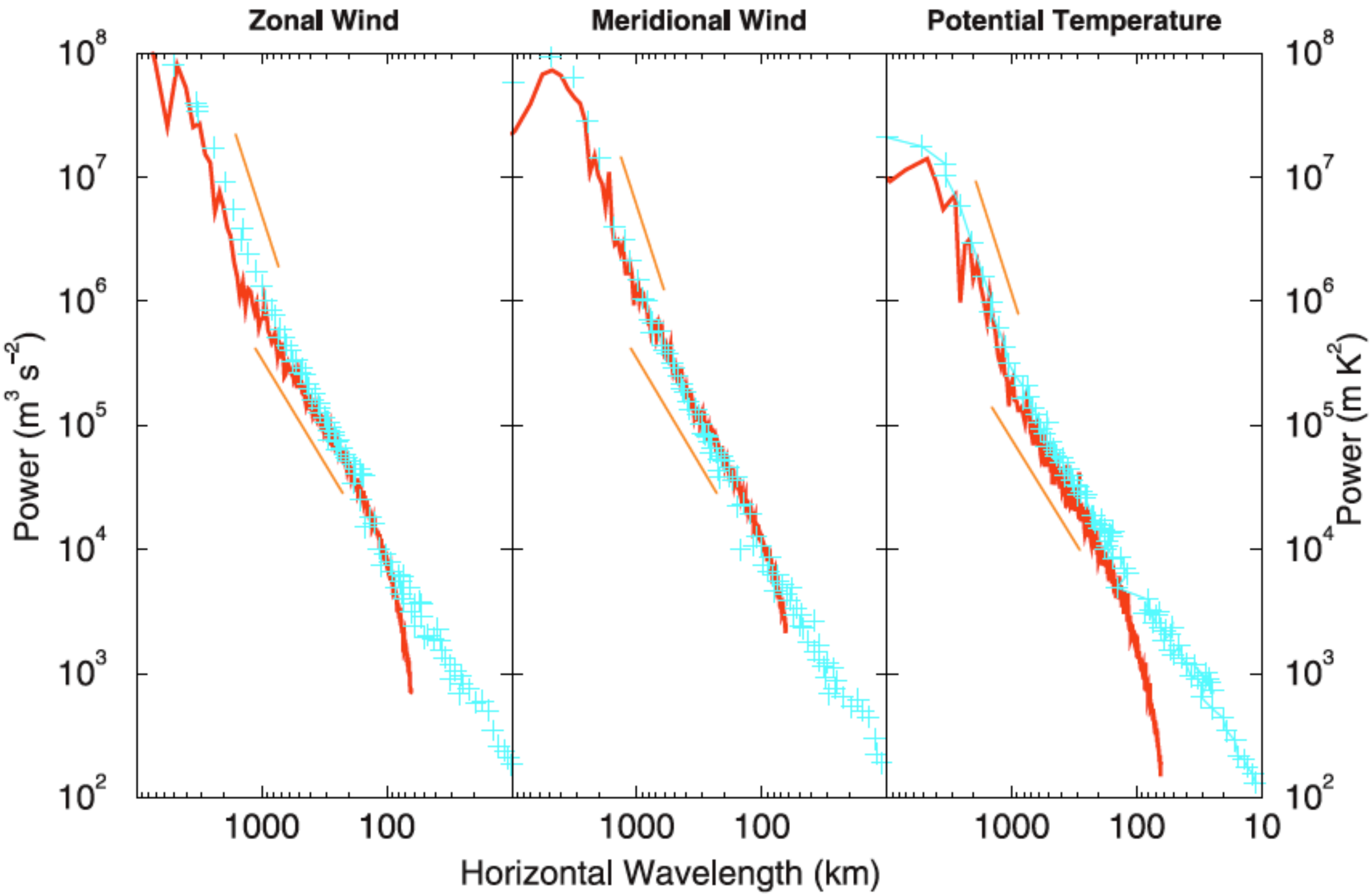


So for the parameterization problem we actually need to understand the dynamics of *this* range

Nastrom & Gage (1985 JAS)

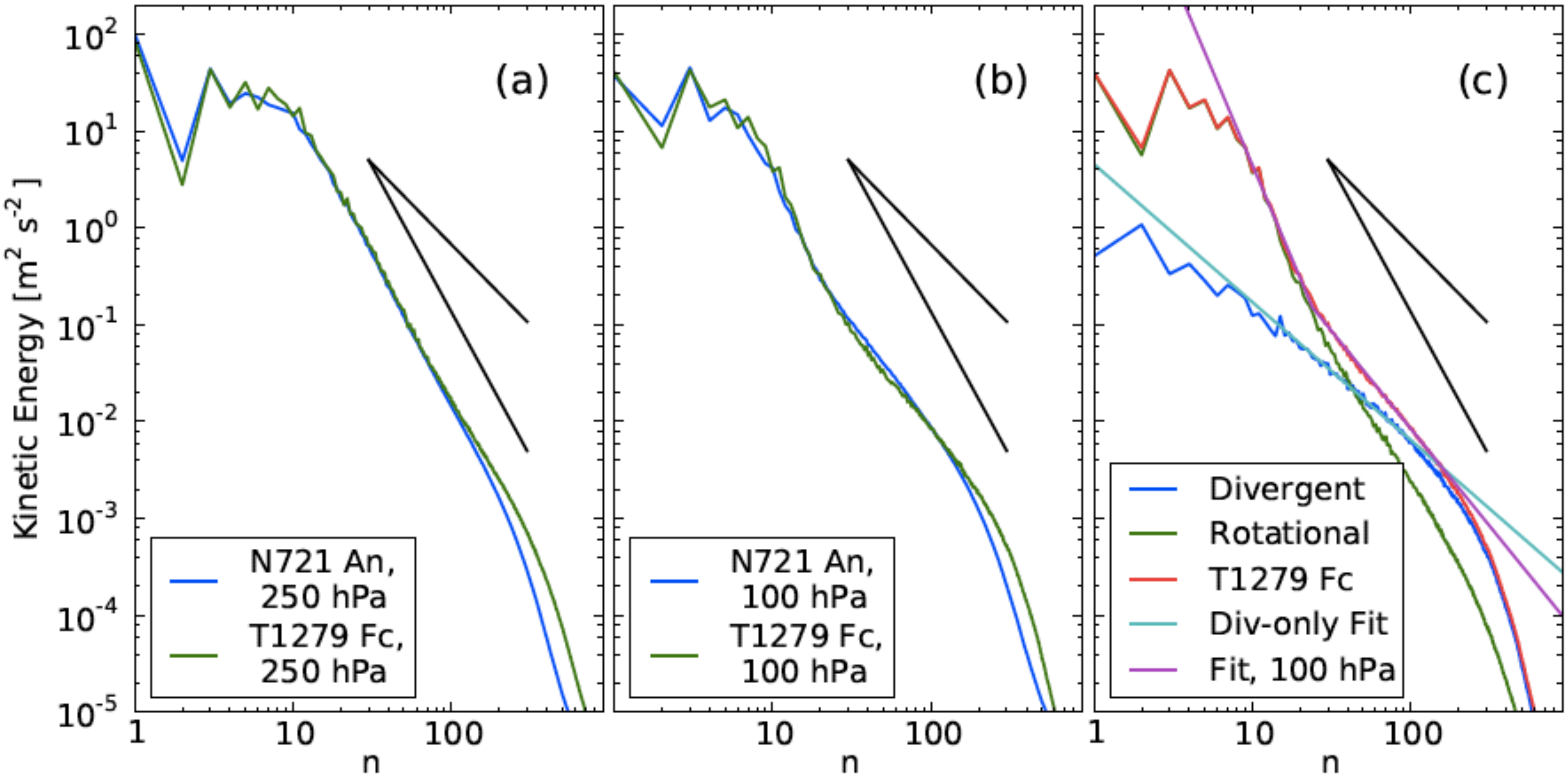
- The origin of the Gage-Nastrom spectrum has been a matter of considerable controversy
  - Some (e.g. Lilly 1983 JAS) have argued for an inverse cascade of balanced (low Froude number) energy from the mesoscale (2-D turbulence)
  - However, evidence appears to be consolidating around a forward (downscale) cascade of unbalanced energy, uninhibited by the potential enstrophy constraint (e.g. Waite & Bartello 2004 JFM; Lindborg 2006 JFM)
    - Imbalance can be generated by a variety of mechanisms
  - One can expect upward radiation of internal gravity waves from any such spectrum
  - There are many ways to get a $k^{-5/3}$ energy spectrum; all one requires is the appropriate scaling symmetry

- The Gage-Nastrom spectrum (blue) is reproduced in high-resolution GCMs (here AFES T639 at 45°N and 200 hPa)



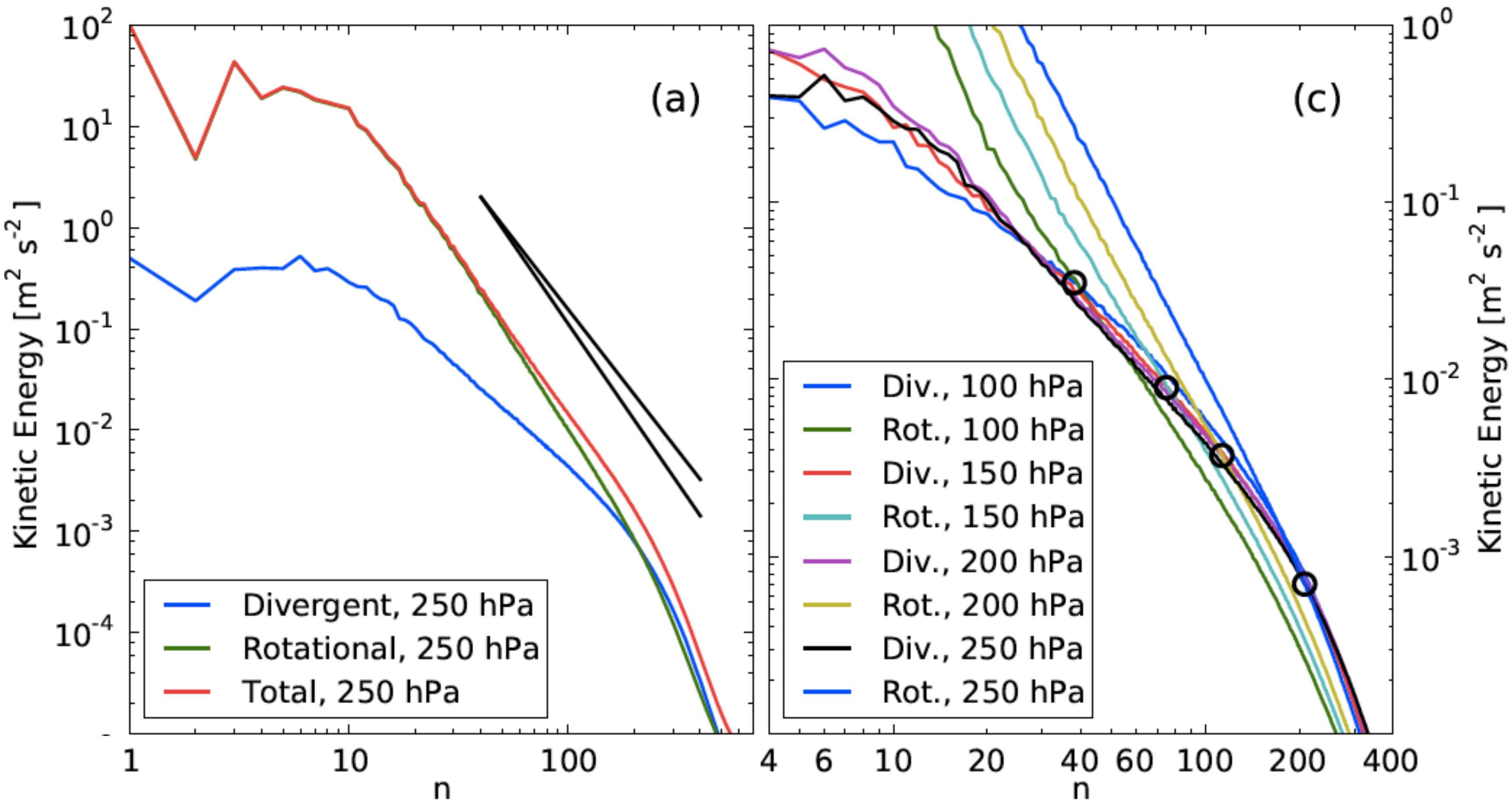
Hamilton, Takahashi & Ohfuchi (2008 JGR)

- It is also seen in ECMWF forecasts at sufficiently high spatial resolution and altitude, and is associated with the emergence of the divergent (unbalanced) component of the spectrum



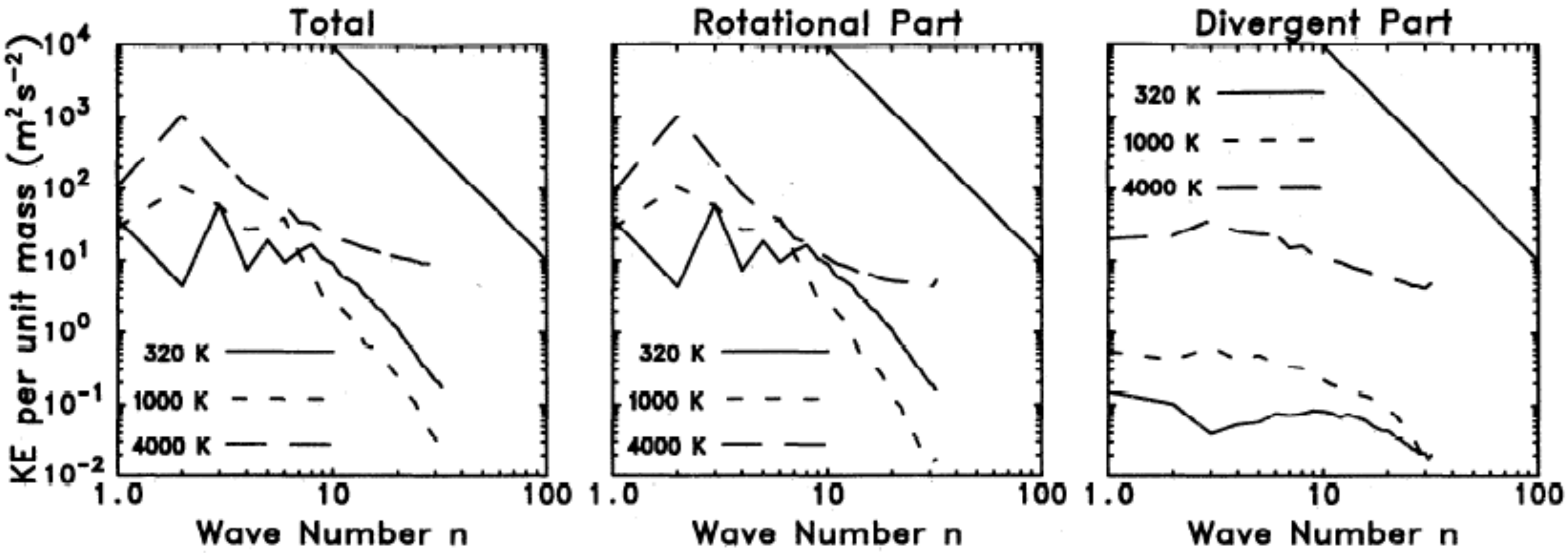
Burgess, Erler & Shepherd (2013 JAS)

- The rotational component of the flow decays with altitude (Charney-Drazin filtering) while the divergent component grows; the spectral break correspondingly moves upscale



Burgess, Erler & Shepherd (2013 JAS)

- Even low-resolution GCMs exhibit an unbalanced spectrum, which emerges at sufficiently high altitudes
  - 320 K isentropic surface is upper troposphere (10 km)
  - 1000 K is middle stratosphere (35 km)
  - 4000 K is middle mesosphere (70 km)



CMAM results from Shepherd, Koshyk & Ngan (2000 JGR)

- Classic paradigm of **atmospheric predictability** (Lorenz 1969 Tellus):
  - Imagine the atmosphere is perfectly observed down to a certain spatial resolution
  - Suppose the forecast model is perfect
  - The initial errors at the smallest scales will eventually contaminate the solution at large scales
  - For how long is the atmosphere predictable?
- Heuristic argument (see Vallis 1985 QJRMS):

Let τ_L be the time for error on horizontal length scale L to introduce error on length scale $2L$. Then the predictability time at scale L, if the initial error is at scale $(1/2)^N L$, is

$$T_N = \tau_{L/2} + \tau_{L/4} + \tau_{L/8} + \ldots + \tau_{L/(2^N)} = \sum_{n=1}^{N} \tau_{\left(\frac{1}{2}\right)^N L}.$$

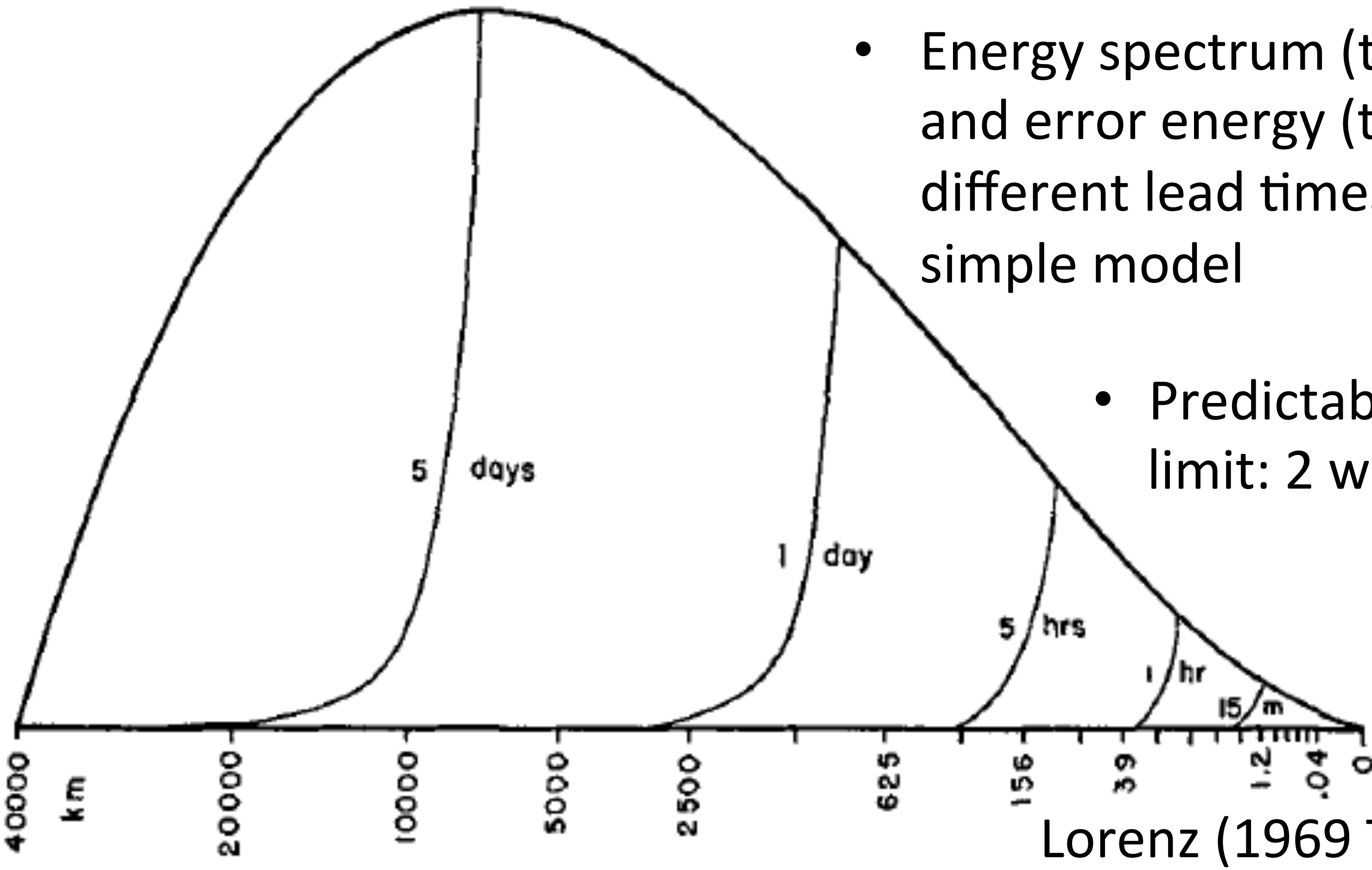
Now, what is τ_L? Dimensional analysis suggests $\tau_L \sim \left(k^3 E(k)\right)^{-0.5}$ where k is the horizontal wavenumber, $E(k)$ is the spectral density of kinetic energy, and $L = \frac{1}{k}$.

$$E(k) \sim k^{-3} \longrightarrow \tau_L \sim \text{const.} \longrightarrow T_N \to \infty \text{ as } N \to \infty$$

$$E(k) \sim k^{-5/3} \longrightarrow \tau_L \sim (k^{4/3})^{-0.5} \sim k^{-2/3} \sim L^{2/3}$$

$$\longrightarrow \lim_{N\to\infty} T_N \sim \sum_{n=1}^{\infty} \left(\frac{L}{2^N}\right)^{2/3} < \infty$$

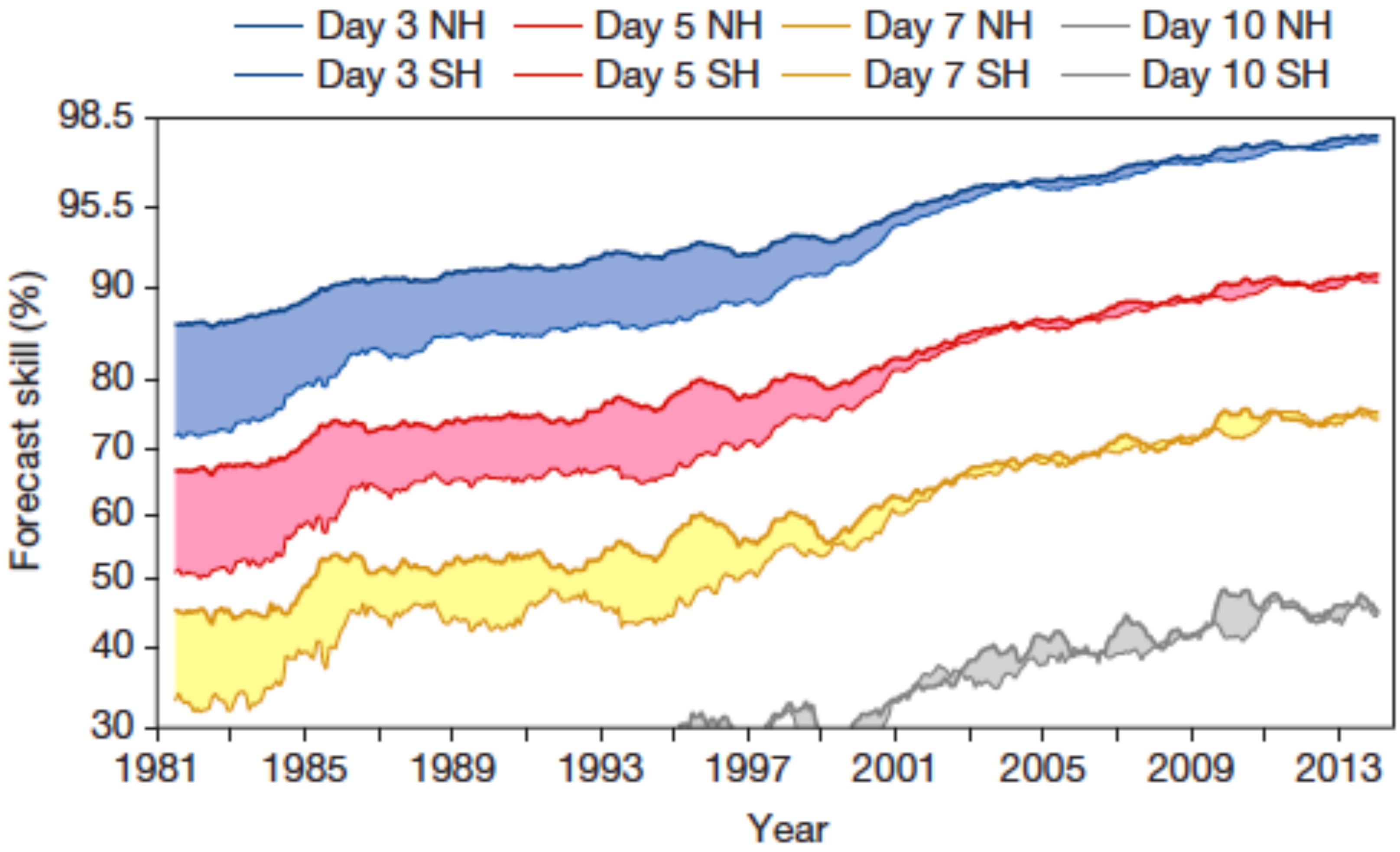
(butterfly effect)



- Energy spectrum (thick) and error energy (thin) at different lead times, in a simple model
  - Predictability limit: 2 weeks!

Lorenz (1969 Tellus)

- Forecast skill in ECMWF system over time (based on mid-tropospheric geopotential height anomalies)



Bauer, Thorpe & Brunet (2015 Nature)