Balanced and unbalanced dynamics in the shallow-water equations

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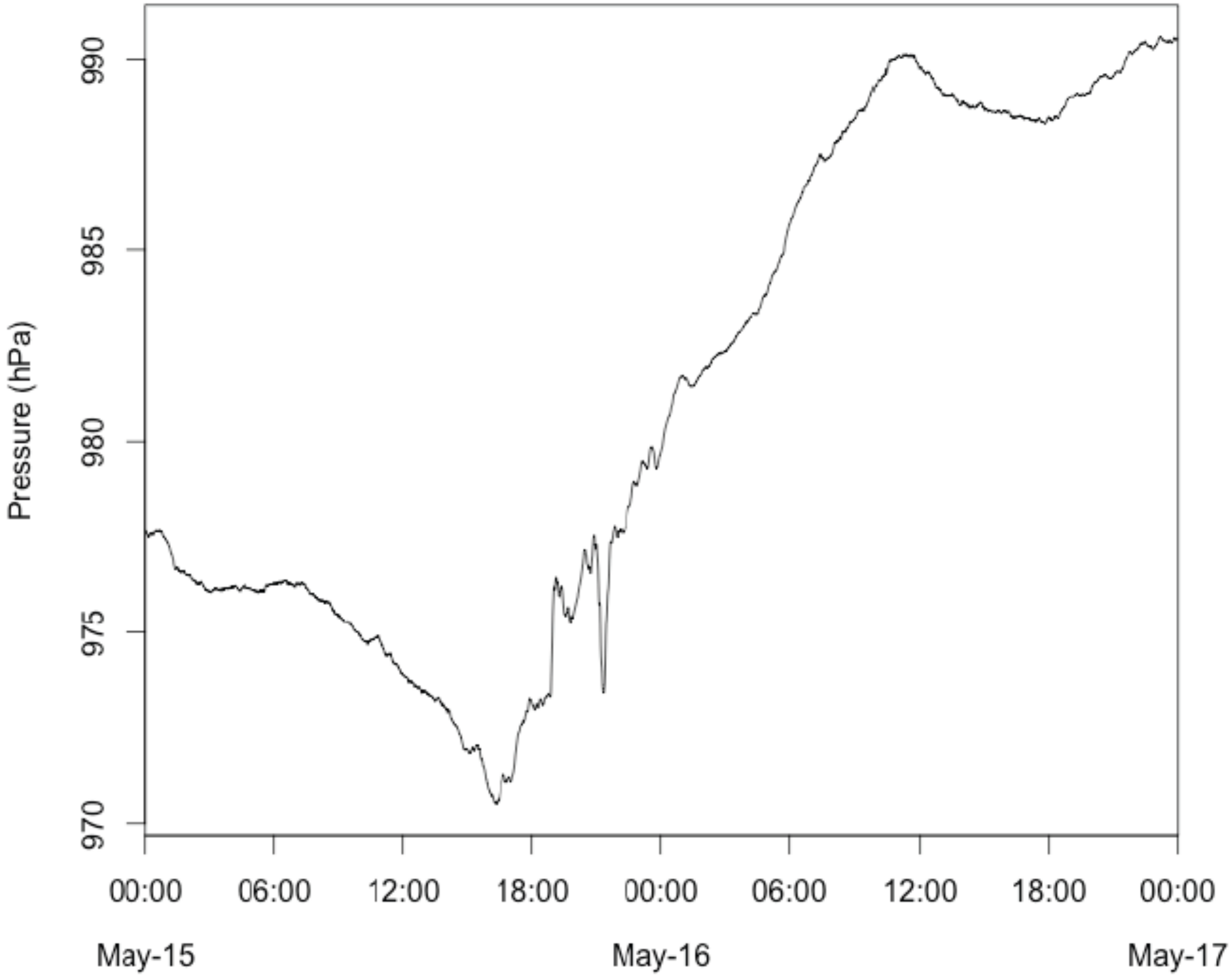
- **Motivation for study of balanced dynamics**

"We might say that the atmosphere is a musical instrument," wrote Charney, "on which one can play many tunes. The high notes are sound waves, low notes are long inertial waves, and nature as a musician is more of a Beethoven than of the Chopin type. She much prefers the low notes, and only occasionally plays arpeggios in the treble, and then only with a light hand.

"The ocean and the continents are the elephants in Saint-Saens's animal suite, marching in a slow cumbrous rhythm, one step every day or so. And, of course, there are overtones - sound waves, billow clouds and inertial oscillations."

Jule Charney
(1917-1981)

- Surface pressure during passage of a cold front and associated squall line at a station in Oklahoma



Data from NOAA

- Leads to reduced models, which are easier to solve (esp. QG)

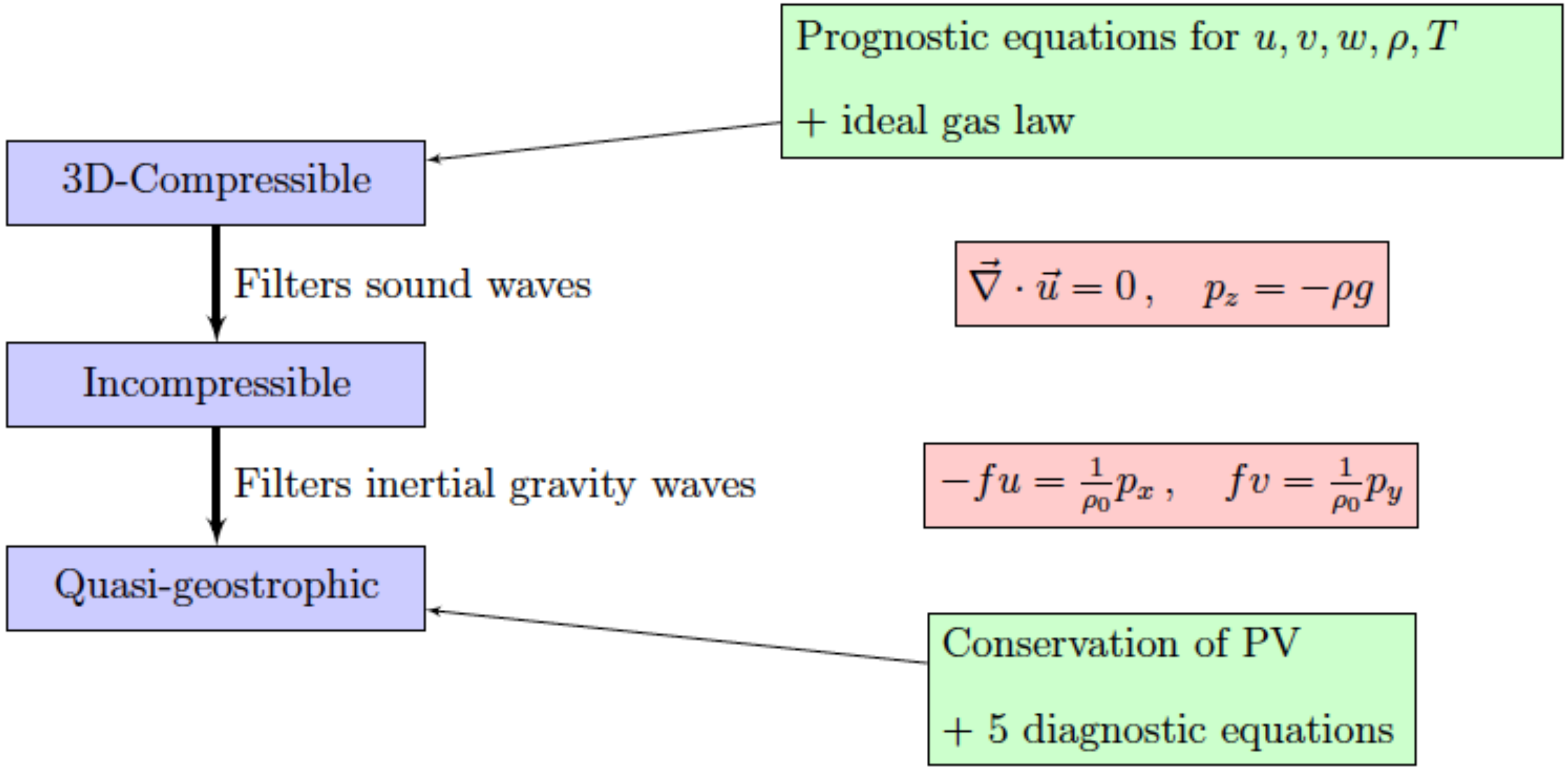


Figure courtesy of Ian Chan

- An arbitrary initial condition will project onto both the fast and the slow degrees of freedom (geostrophic adjustment)

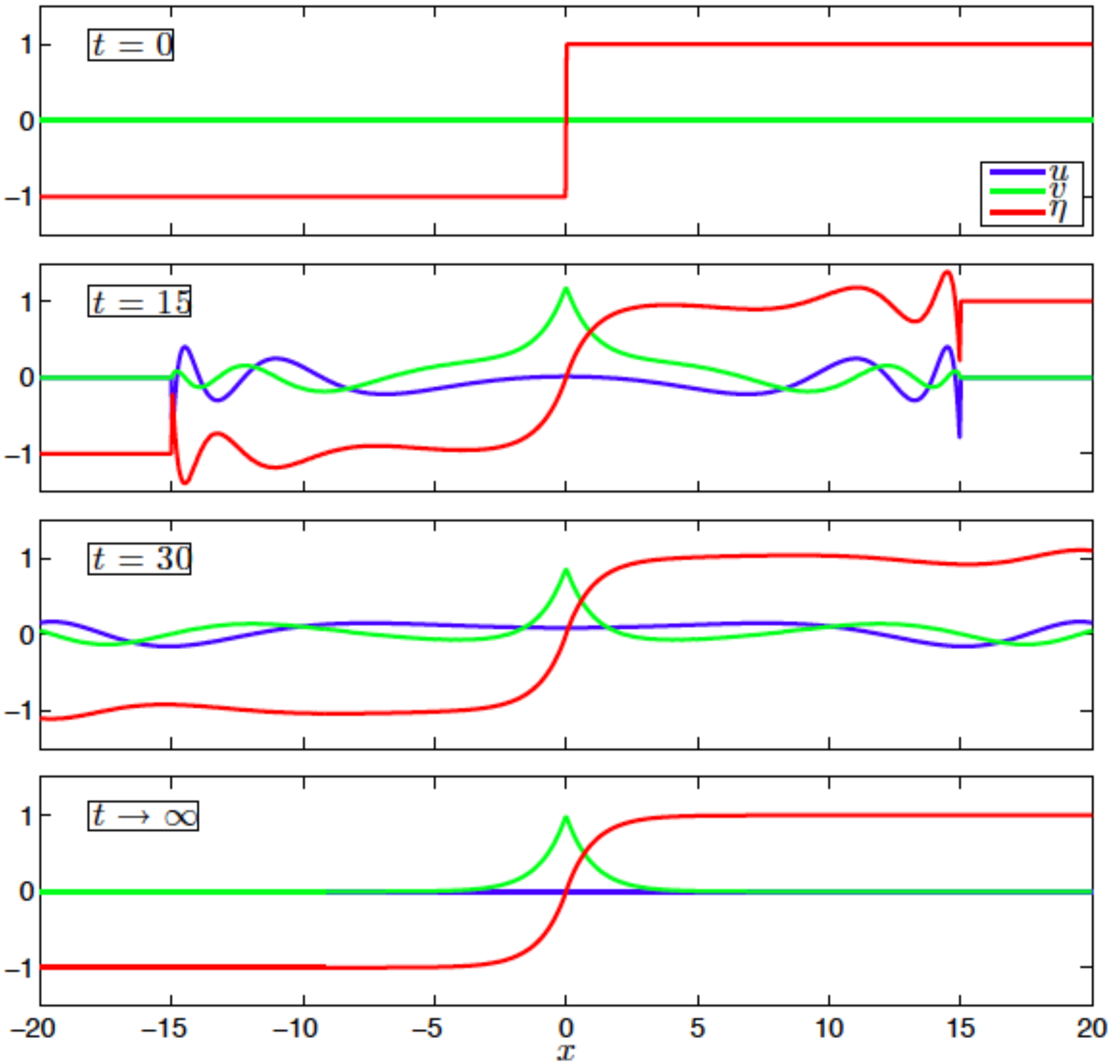
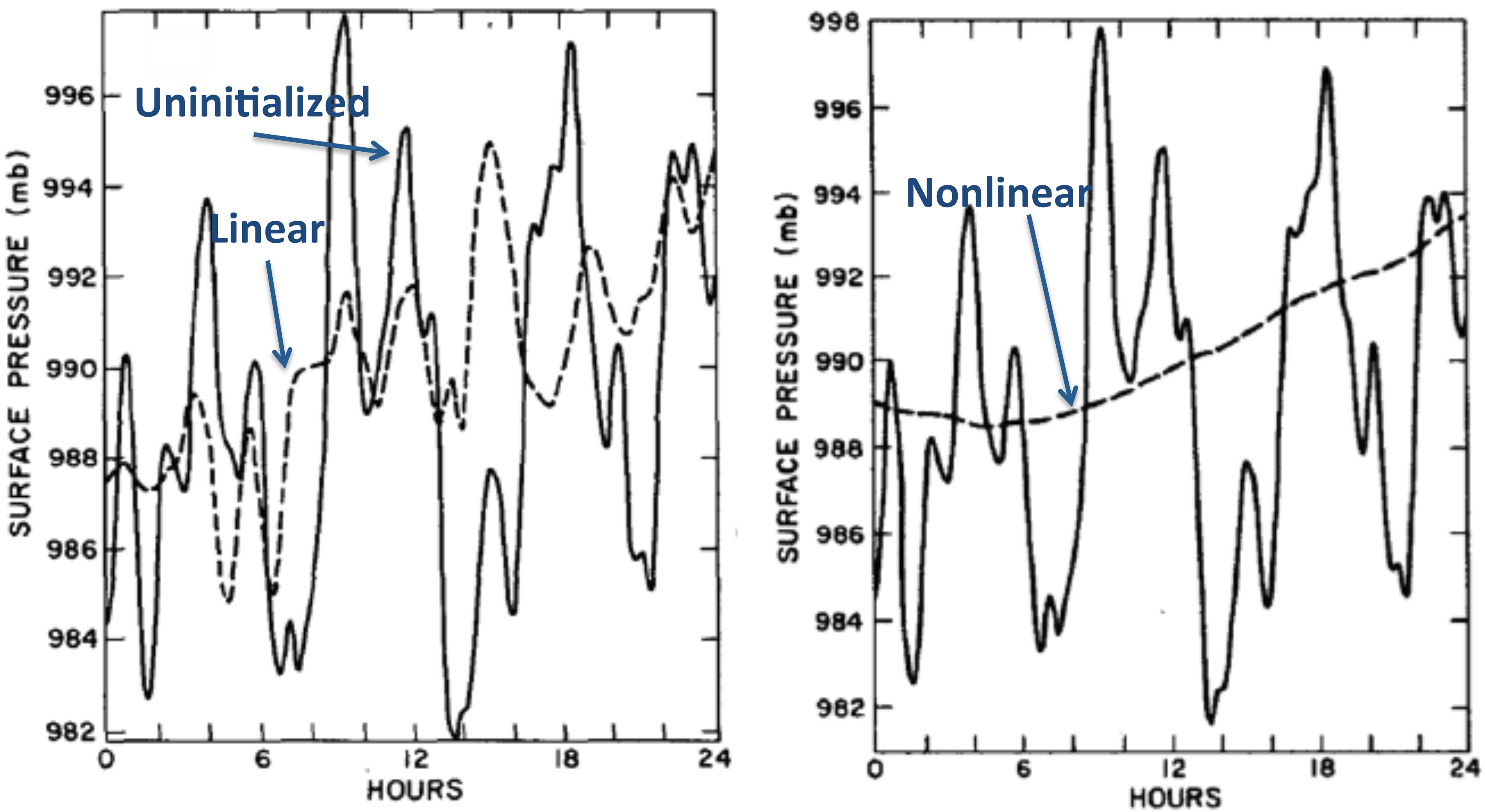


Figure courtesy of Ian Chan

- Observed initial conditions will have a spuriously large projection onto IG waves, will generate unphysical oscillations
- Initialization can control these oscillations (slow manifold)



Temperton & Williamson (1981)

- Consider the **shallow-water equations on the *f*-plane** (a very important model in atmosphere-ocean dynamics!)
  - Analogous to horizontal dynamics of hydrostatic primitive equations in isentropic coordinates

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + f\,\hat{z} \times \vec{u} = -g\nabla h, \quad \frac{\partial h}{\partial t} + \nabla \cdot (h\vec{u}) = 0$$

- Linearize about a state of rest with constant depth $h = H$

  Then the ansatz $\exp\{i(kx + \ell y - \omega t)\}$ leads to

  $\omega_0 = 0$ or $\omega_{IG}^2 = f^2 + gH\kappa^2$ (IG waves), where $\kappa^2 = k^2 + \ell^2$

- Depending on κ, the IG waves may be dominated either by rotation or by gravity (surface waves)
- Note that $|\omega_{IG}| > f$ (low-frequency cut-off because of rotation)

- Introduce a Doppler shift $U\kappa$ to the frequency to take account of the nonlinear advection term in the dynamics. Then

$$\frac{\omega_0}{\omega_{IG}} \approx \frac{U\kappa}{U\kappa + \sqrt{f^2 + gH\kappa^2}} << 1 \quad \Leftrightarrow \quad \varepsilon = \frac{U\kappa}{\sqrt{f^2 + gH\kappa^2}} << 1$$

- Under these conditions we have a **separation in timescales**, and can distinguish between **fast and slow dynamics**
- Define $Ro = \frac{U\kappa}{f}$ (Rossby no.) and $Fr = \frac{U}{\sqrt{gH}}$ (Froude no.)

(note analogue of *Fr* for continuously stratified flow). Then

$$\varepsilon = \frac{Ro\ Fr}{\sqrt{Ro^2 + Fr^2}}$$

- Hence ε << 1 if either *Ro* or *Fr* are small, even if the other parameter diverges (Saujani & Shepherd 2006 *JFM*)

- The ratio *Fr*/*Ro* determines whether rotation or gravity dominates

$$\frac{Fr}{Ro} = \frac{f}{\kappa\sqrt{gH}} = \frac{L}{L_R} \; ; \quad L = 1/\kappa,\ L_R = \text{Rossby deformation radius}$$

- In classical **quasi-geostrophic scaling**, $Ro = Fr$ and the effects of rotation and gravity are taken to be comparable
- Now **non-dimensionalize** the full shallow-water equations with characteristic velocity scale U, horizontal length scale L_R, the **slow** advective time scale L_R/U, and $h = H\,(1 + Ro\,\eta)$
- This gives the dimensionless system

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} + \frac{1}{Ro}\left(\hat{z}\times\vec{u} + \nabla\eta\right) = 0, \quad \frac{\partial \eta}{\partial t} + \nabla\cdot\,(\eta\vec{u}) + \frac{1}{Ro}\nabla\cdot\vec{u} = 0$$

- If $Ro << 1$, then to obtain slow motion we require…

$$\hat{z} \times \vec{u} + \nabla \eta = 0, \quad \nabla \cdot \vec{u} = 0, \quad i.e. \quad u = -\frac{\partial \eta}{\partial y}, \; v = \frac{\partial \eta}{\partial x}$$ (geostrophic balance)

- Equivalently, $\psi = \eta$ and $\chi = 0$ where $\vec{u} = \hat{z} \times \nabla \psi + \nabla \chi$
- But there are actually only two independent constraints here; eliminating the 1/*Ro* terms in the equations gives

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta\right) = 0$$ (**QG potential vorticity equation**)

- This is now a closed, first-order time-evolution equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0 \quad \text{where} \quad q = \nabla^2 \psi - \psi$$

(Note requirement of additional boundary condition on the circulation along lateral sidewalls)

- The existence of a single slow equation is no accident, since we had a slow (zero-frequency) linear mode in the system

- Note that (as with incompressibility or hydrostatic balance) the reduction is not exact; but an asymptotic procedure can be developed to define an approximate **slow manifold**, with the "fast" variables *f* slaved to the "slow" variables *s* (Warn et al. 1995 *QJRMS*)

$$\frac{\partial f}{\partial t} + \frac{\Gamma f}{\varepsilon} = F(s, f; \varepsilon), \quad \frac{\partial s}{\partial t} = S(s, f; \varepsilon)$$

$$f = U(s;\varepsilon) \quad \Rightarrow \quad f = f^{(0)} + \varepsilon f^{(1)} + \ldots$$

$$O(1): \quad f^{(0)} = 0, \quad \frac{\partial s}{\partial t} = S(s,0;0)$$

$$O(\varepsilon): \quad f^{(1)} = \Gamma^{-1} F(s,0;0), \quad \frac{\partial s}{\partial t} = S(s, f^{(1)};\varepsilon)\Big|_{O(\varepsilon)}$$

and so on (Bokhove & Shepherd 1996 *J.Atmos.Sci.* show the asymptotic nature of this series for a low-order ODE system)

- In geophysical fluid dynamics, a slow equation is always provided by the **potential vorticity**, which necessarily evolves on the advective timescale (Hoskins et al. 1985 *QJRMS*)
- In the special case of the shallow-water equations,

$$Q = \frac{f + \hat{z}\cdot \nabla \times \vec{u}}{h} \;\rightarrow\; \frac{f(1+Ro\,\hat{z}\cdot \nabla \times \vec{u})}{H(1+Ro\,\eta)}$$

$$\approx \frac{f}{H}\left[1 + Ro\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta\right) + \ldots\right]$$

which apart from constants is the QG potential vorticity *q*,

and the fast variables are $\delta = \nabla\cdot \vec{u} = \nabla^2\chi$ and $\nabla^2(\psi - \eta)$

- The slaving relations are known as **balance relations**, and the slow dynamics is known as **balanced dynamics**
  – O(1) is QG balance, O(ε) is Charney-Bolin balance

- **Potential vorticity slaving in the shallow-water system**

$$\frac{\partial q}{\partial t} = -J(\psi, q) - \nabla\chi \cdot \nabla q,$$

$$\frac{\partial D}{\partial t} - \frac{\Omega}{\varepsilon} = -J(\chi, \nabla^2\psi) - \frac{1}{2}\nabla^2|\nabla\chi|^2 + \nabla^2 J(\chi, \psi) + 2J(\psi_x, \psi_y)$$

$$\frac{\partial \Omega}{\partial t} - \frac{\mathcal{H}D}{\varepsilon} = -J(\psi, \nabla^2\psi) - \nabla\cdot(\nabla^2\psi\,\nabla\chi) + \nabla^2\{J(\psi, \eta) + \nabla\cdot(\eta\,\nabla\chi)\}$$

$$1 + \varepsilon q = \frac{1 + \varepsilon\nabla^2\psi}{1 + \varepsilon\eta} \iff q = \frac{\nabla^2\psi - \eta}{1 + \varepsilon\eta}$$

$$D = \nabla\cdot\mathbf{v} = \nabla^2\chi \qquad \Omega = \nabla^2\psi - \nabla^2\eta$$

$$\mathcal{H} = \nabla^2 - 1 \qquad \mathbf{v} = \mathbf{k}\times\nabla\psi + \nabla\chi$$

Warn et al. (1995 QJRMS)

O(1): $\chi^{(0)} = 0$ $\Omega^{(0)} = 0$ $\nabla^2\psi^{(0)} - \psi^{(0)} = q$

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0$$

O(ε): $\Omega^{(1)} = -2J(\psi_x^{(0)}, \psi_y^{(0)})$ $\Lambda\chi^{(1)} = J(\psi^{(0)}, \nabla^2\psi^{(0)})$

$$\nabla^2\psi^{(1)} - \eta^{(1)} = \eta^{(0)}q$$

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0 \quad \psi = \psi^{(0)} + \varepsilon\psi^{(1)} \quad \chi = \chi^{(0)} + \varepsilon\chi^{(1)}$$

$$\Lambda\chi = \varepsilon J(\gamma, \nabla^2\gamma)$$

$\Lambda \equiv \mathcal{H}\nabla^2$ $\gamma \equiv \psi^{(0)} = \mathcal{H}^{-1}q$

$$\Lambda\psi = \nabla^2 q + \varepsilon\{\nabla^2(\gamma q) + 2J(\gamma_x, \gamma_y)\}$$

Warn et al. (1995 QJRMS)

O(ε^2):

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0$$

$$\Lambda\psi = \nabla^2 q + \varepsilon R(\gamma) + \varepsilon^2 P(\gamma) \qquad \Lambda\chi = \varepsilon G(\gamma) + \varepsilon^2 C(\gamma)$$

$$G(\gamma) = J(\gamma, \nabla^2\gamma)$$

$$R(\gamma) = \nabla^2(\gamma q) + 2J(\gamma_x, \gamma_y)$$

$$\begin{aligned} C(\gamma) = {} & 2J(\partial_x\{\mathcal{H}^{-1}J(\gamma, q)\}, \gamma_y) + 2J(\gamma_x, \partial_y\{\mathcal{H}^{-1}J(\gamma, q)\}) + J(\Lambda^{-1}R(\gamma), \nabla^2\gamma) + \\ & + J(\gamma, \nabla^2\Lambda^{-1}R(\gamma)) + \nabla\cdot\{\nabla^2\gamma\nabla(\Lambda^{-1}G(\gamma))\} - \\ & - \nabla^2[J(\gamma, \mathcal{H}^{-1}R(\gamma) - \gamma q) + J(\Lambda^{-1}R(\gamma), \gamma) + \nabla\cdot\{\gamma\nabla(\Lambda^{-1}G(\gamma))\}] \end{aligned}$$

$$\begin{aligned} P(\gamma) = {} & \nabla^2\{(\mathcal{H}^{-1}R(\gamma) - \gamma q)q\} + \mathcal{H}^{-1}J(\mathcal{H}^{-1}J(\gamma, q), \nabla^2\gamma) + \\ & + \mathcal{H}^{-1}J(\gamma, \nabla^2\mathcal{H}^{-1}J(\gamma, q)) - J(\Lambda^{-1}G(\gamma), \nabla^2\gamma) + \nabla^2 J(\Lambda^{-1}G(\gamma), \gamma) + \\ & + 2J(\partial_x(\Lambda^{-1}R(\gamma)), \gamma_y) + 2J(\gamma_x, \partial_y(\Lambda^{-1}R(\gamma))), \end{aligned}$$

Warn et al. (1995 QJRMS)

On the Existence of a Slow Manifold

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(Manuscript received and in final form 28 October 1985)

On the Nonexistence of a Slow Manifold

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(Manuscript received 29 September 1986, in final form 13 April 1987)

The Slow Manifold—What Is It?

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(Manuscript received 17 June 1991, in final form 17 March 1992)

- **The Lorenz (1986 JAS) 5-component model**
- Turns out to be a Hamiltonian system, with invariants

$$H' = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2) \qquad C = \tfrac{1}{2}(x_1^2 + x_2^2)$$

- Integrable for *b*=0 (pendulum coupled to harmonic oscillator)

$$\frac{dx_1}{dt} = -x_2x_3 + bx_2x_5,$$

$$\frac{dx_2}{dt} = x_1x_3 - bx_1x_5,$$

$$\frac{dx_3}{dt} = -x_1x_2,$$

$$\frac{dx_4}{dt} = -\frac{x_5}{\epsilon},$$

$$\frac{dx_5}{dt} = \frac{x_4}{\epsilon} + bx_1x_2.$$

$$x_1 = \sqrt{2C}\cos\phi,\ x_2 = \sqrt{2C}\sin\phi$$

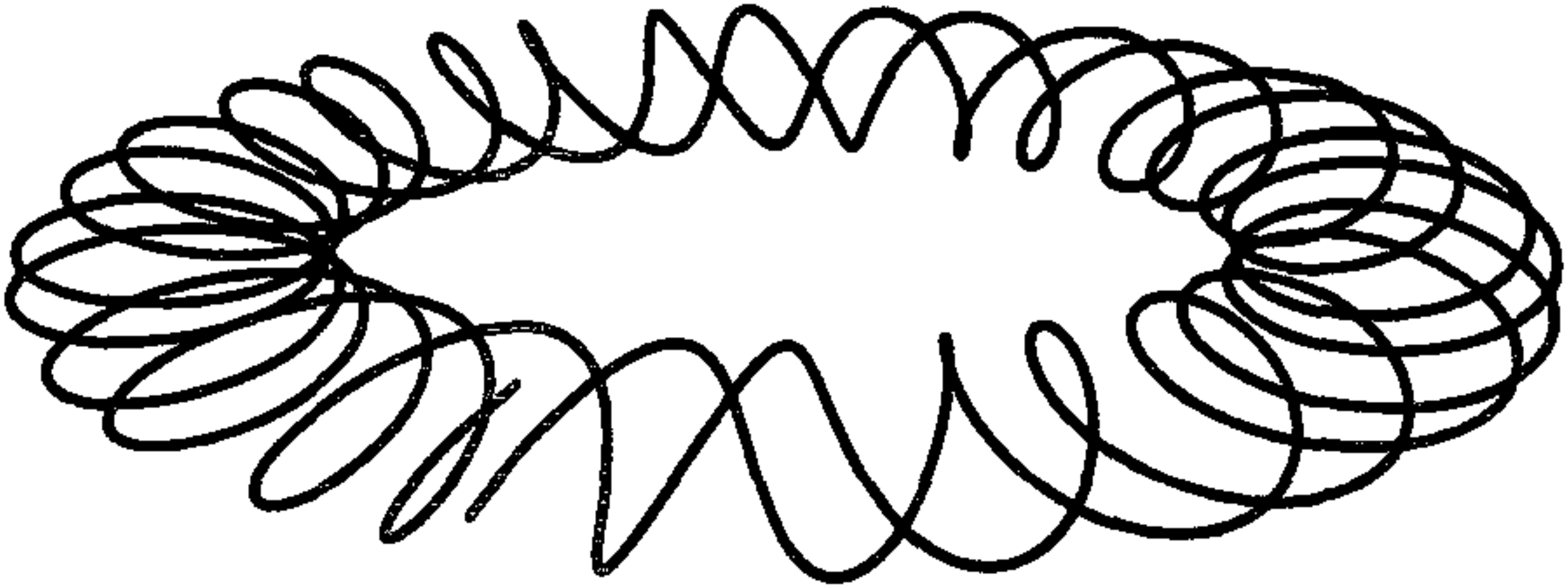
$$\frac{d\phi}{dt} = x_3 - bx_5,$$

$$\frac{dx_3}{dt} = -C\sin 2\phi,$$

$$\frac{dx_4}{dt} = -\frac{x_5}{\epsilon},$$

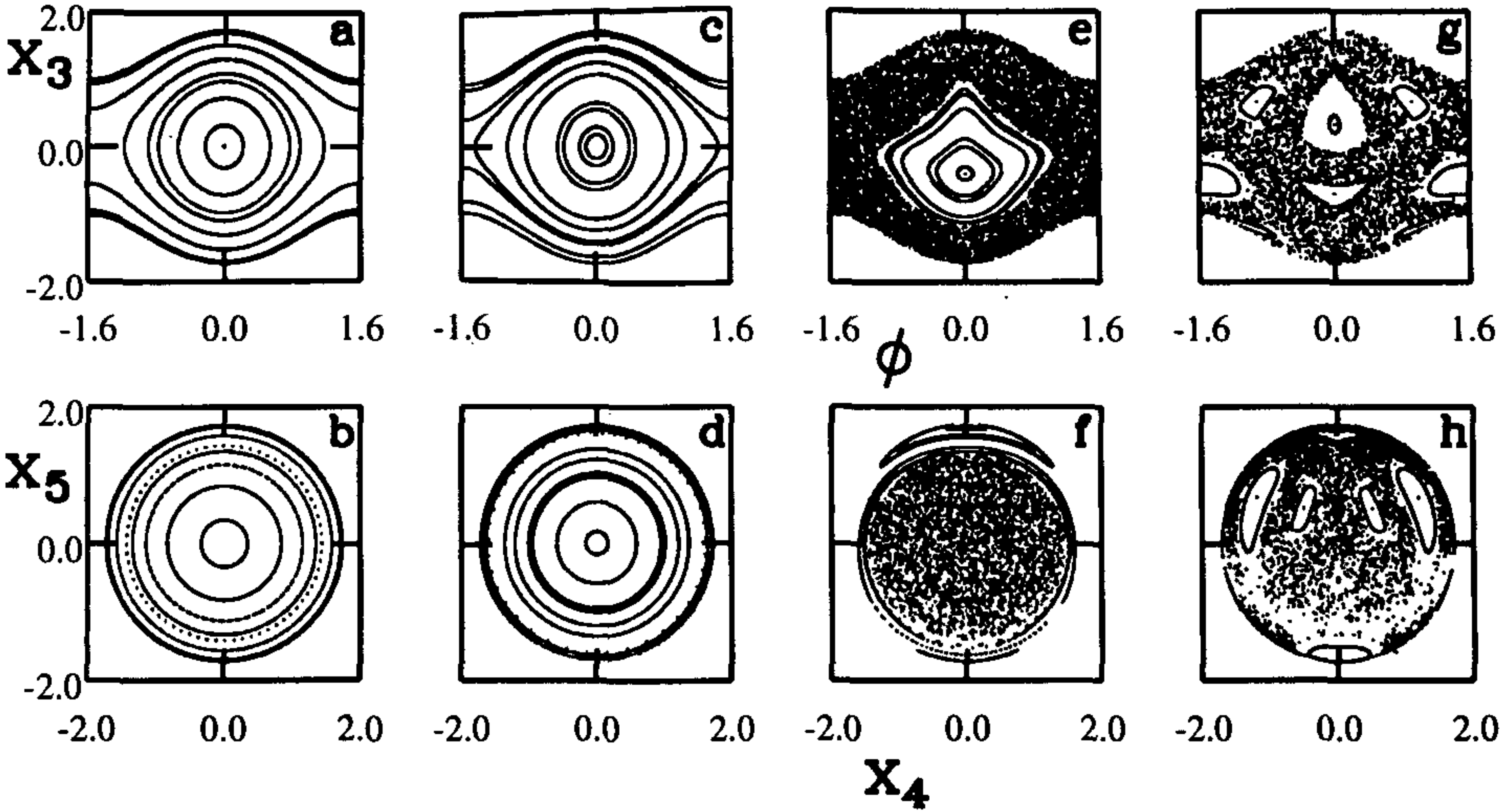
$$\frac{dx_5}{dt} = \frac{x_4}{\epsilon} + bC\sin 2\phi.$$

- In the integrable case, solutions live on a two-torus, and generally are quasi-periodic (not periodic)

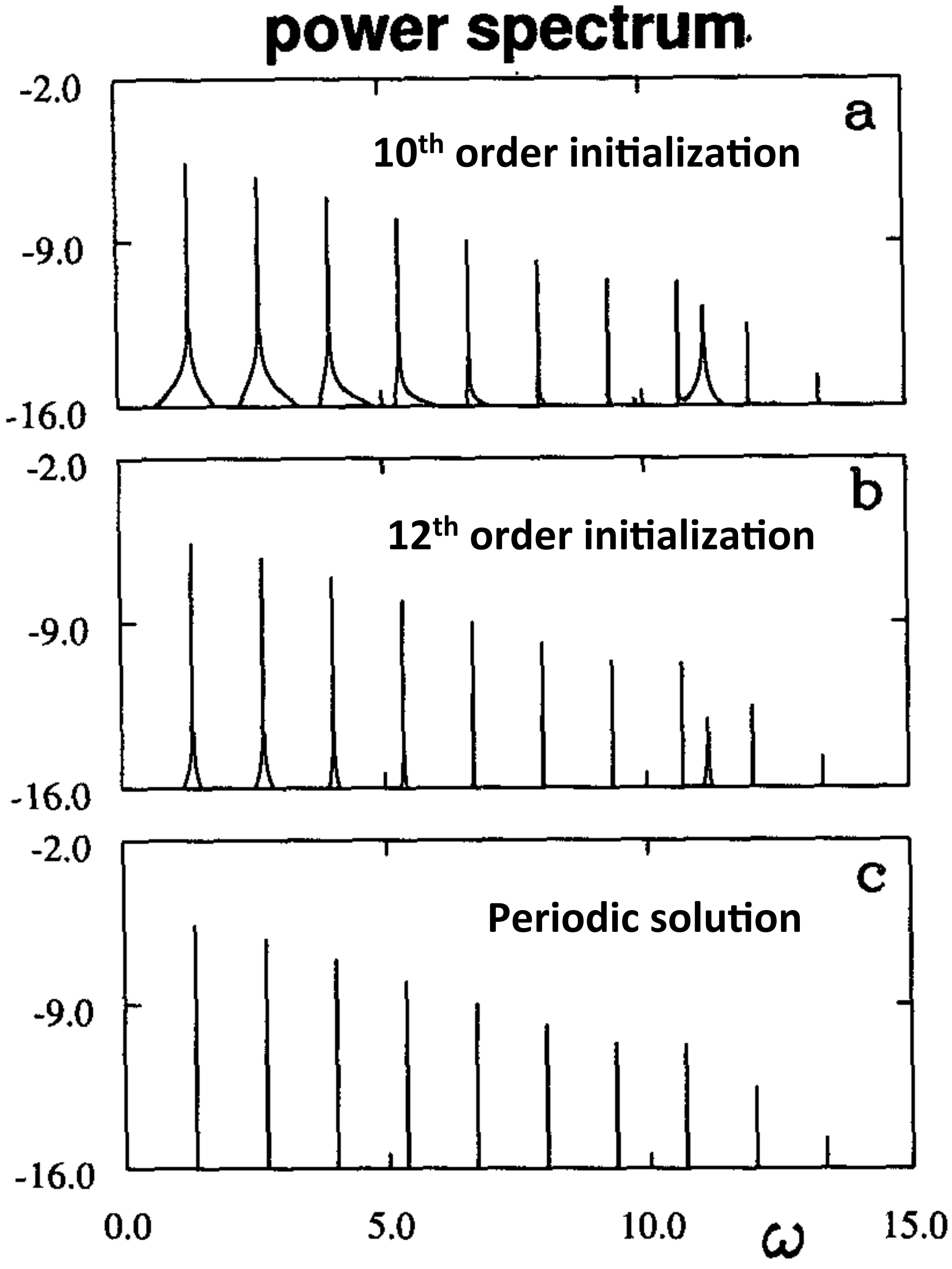


Bokhove & Shepherd (1996 JAS)

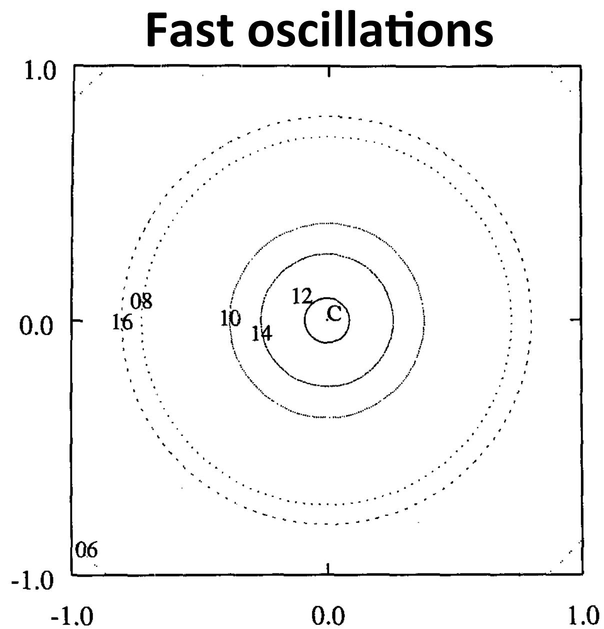
- KAM theorem applies: most invariant tori preserved for sufficiently small perturbations *b*
- Poincaré sections: *b* increasing from zero (left)



Bokhove & Shepherd (1996 JAS)

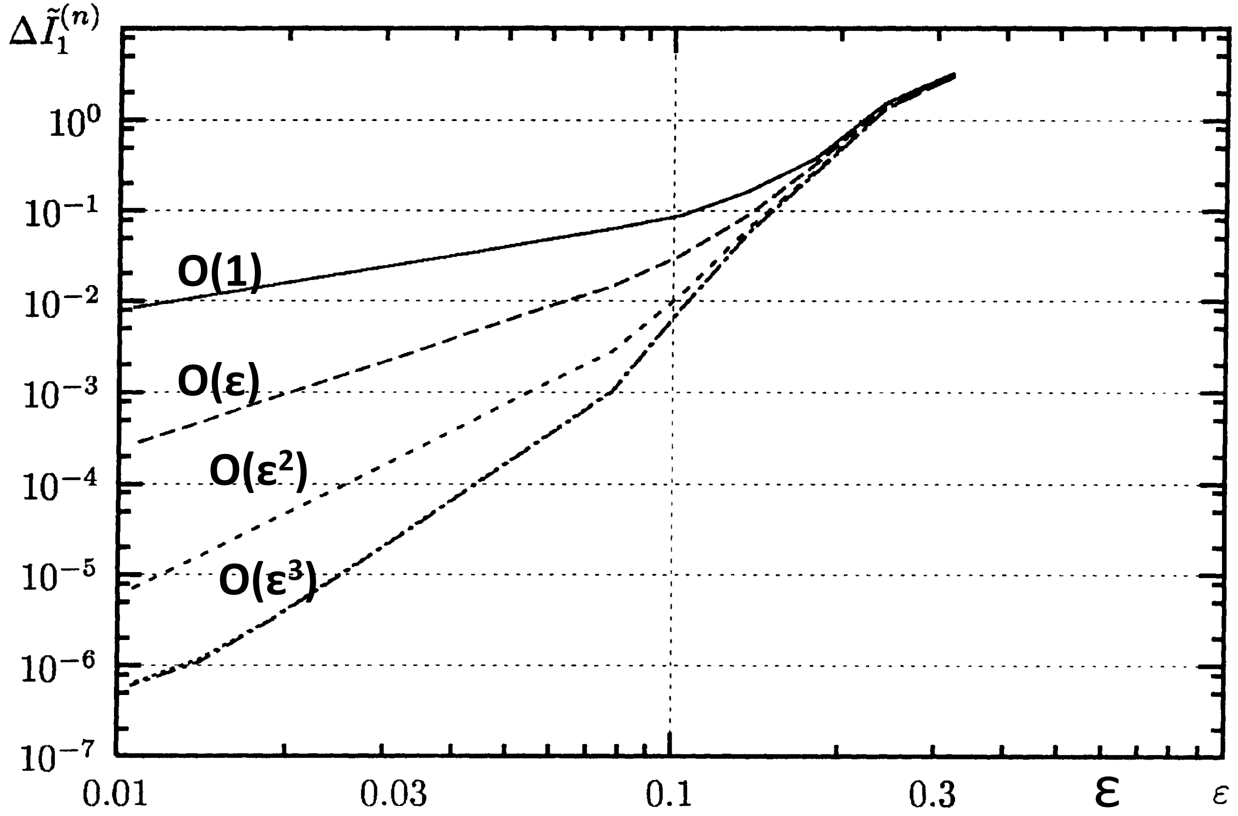


- An exact slow manifold can be defined geometrically
- Slaving approximation is only asymptotic
- Lorenz (1986) found the slow solutions as periodic solutions



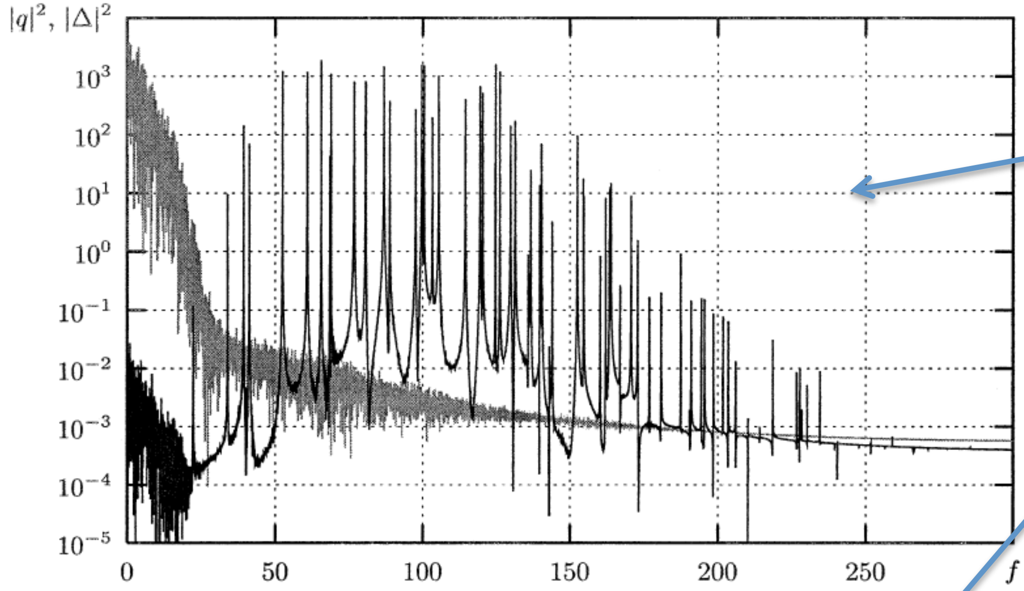
Bokhove & Shepherd (1996 JAS)

- Lorenz (1986) system is pathological because slow dynamics is integrable: unambiguous separation of fast and slow dynamics
- Making *C* periodic in time makes the slow dynamics chaotic
- Fast action is still bounded for finite times by adiabatic invariance (Nekhoroshev 1971, Neishtadt 1984)

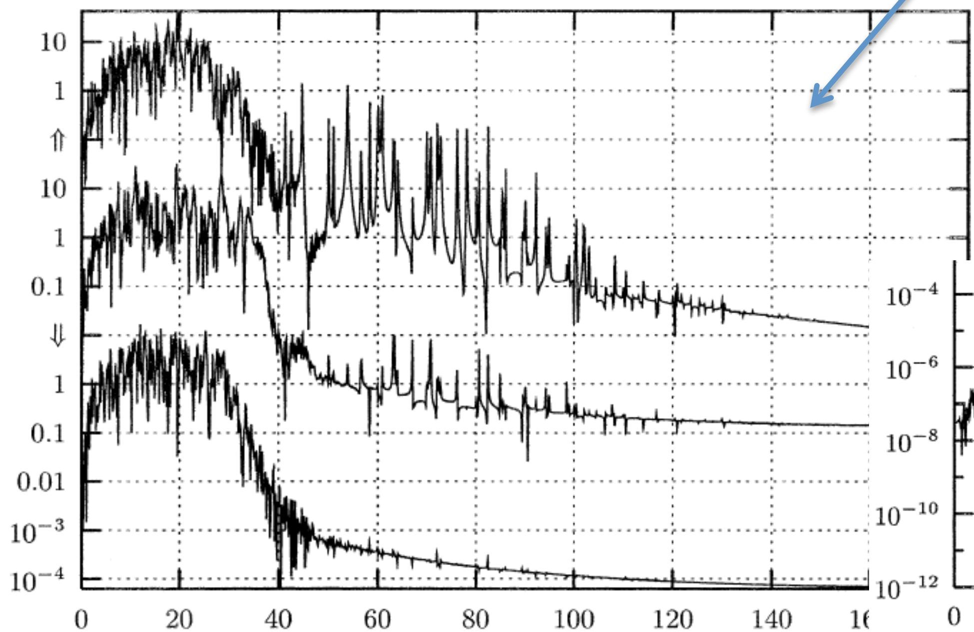


Wirosoetisno & Shepherd (2000 Physica D)

A yet more complex model

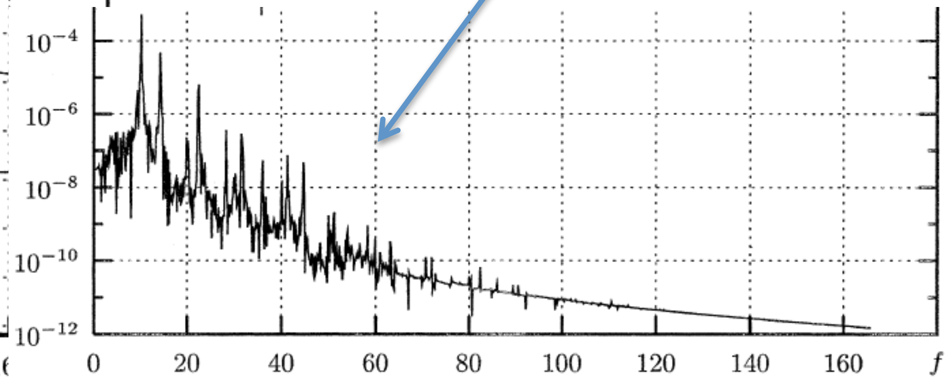


Power spectra of potential vorticity and divergence (minimum frequency is 22)



Power spectra of divergence for O(1), O(ε) and O(ε³) initialization (slaving)

Power spectrum of unslaved divergence in the last case (minimum frequency is 10)

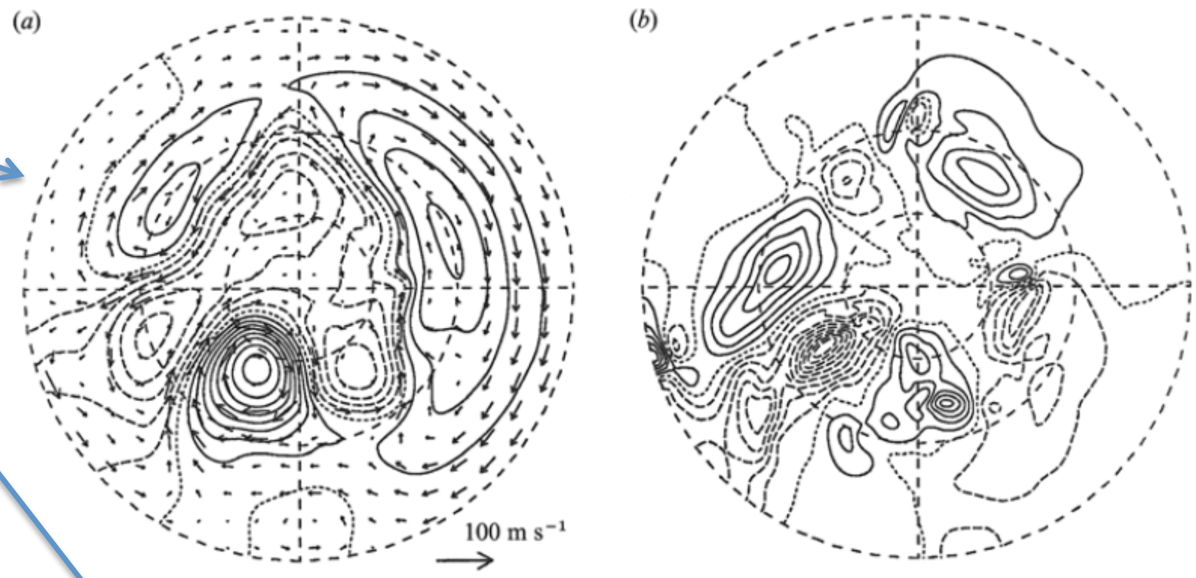


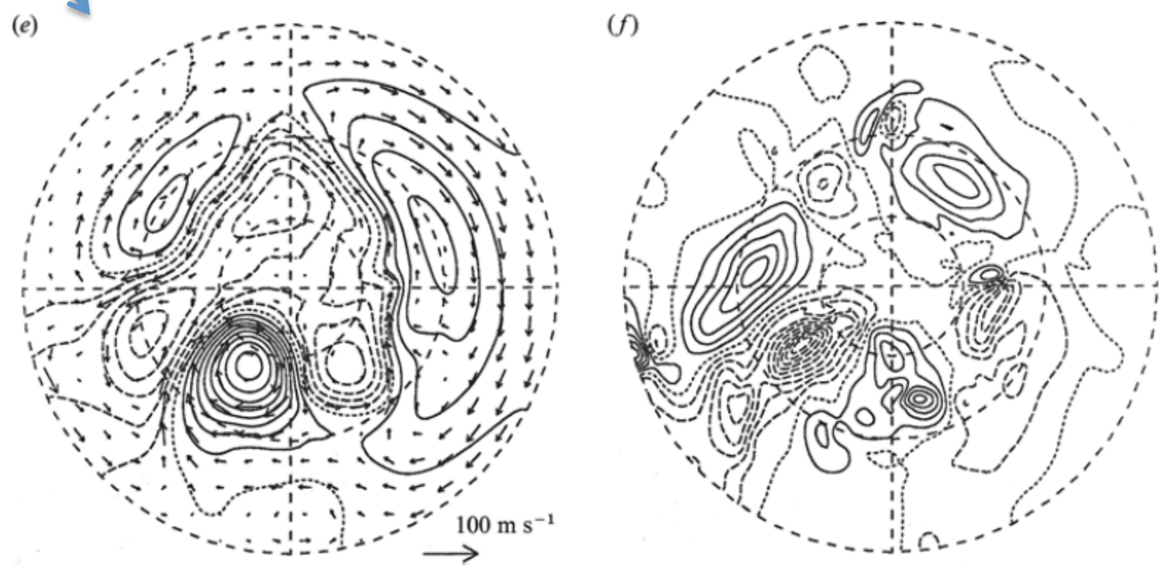
Wirosoetisno, Shepherd & Temam (2002 *J.Atmos.Sci.*)

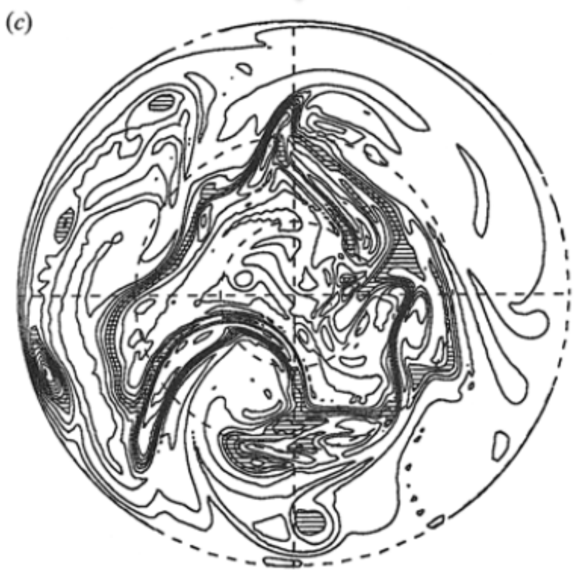
Instantaneous height/velocity and divergence fields

The same, but derived solely from potential vorticity

The instantaneous potential vorticity field







McIntyre & Norton (2000 *J.Atmos.Sci.*)