Balanced and unbalanced dynamics in the shallow-water equations

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Motivation for study of balanced dynamics

"We might say that the atmosphere is a musical instrument," wrote Charney, "on which one can play many tunes. The high notes are sound waves, low notes are long inertial waves, and nature as a musician is more of a Beethoven than of the Chopin type. She much prefers the low notes, and only occasionally plays arpeggios in the treble, and then only with a light hand.

"The ocean and the continents are the elephants in Saint-Saens's animal suite, marching in a slow cumbersome rhythm, one step every day or so. And, of course, there are overtones - sound waves, billow clouds and inertial oscillations."

Jule Charney (1917-1981)
• Surface pressure during passage of a cold front and associated squall line at a station in Oklahoma

Data from NOAA
• Leads to reduced models, which are easier to solve (esp. QG)

Prognostic equations for $u, v, w, \rho, T$
+ ideal gas law

\[ \nabla \cdot \vec{u} = 0, \quad p_z = -\rho g \]

\[-fu = \frac{1}{\rho_0} p_x, \quad fv = \frac{1}{\rho_0} p_y \]

Conservation of PV
+ 5 diagnostic equations

Figure courtesy of Ian Chan
• An arbitrary initial condition will project onto both the fast and the slow degrees of freedom (geostrophic adjustment)

Figure courtesy of Ian Chan
• Observed initial conditions will have a spuriously large projection onto IG waves, will generate unphysical oscillations.
• Initialization can control these oscillations (slow manifold).

Temperton & Williamson (1981)
• Consider the **shallow-water equations on the $f$-plane** (a very important model in atmosphere-ocean dynamics!)
  – Analogous to horizontal dynamics of hydrostatic primitive equations in isentropic coordinates

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{u} + f \hat{z} \times \tilde{u} = -g \nabla h, \quad \frac{\partial h}{\partial t} + \nabla \cdot (h \tilde{u}) = 0$$

• Linearize about a state of rest with constant depth $h = H$
  Then the ansatz $\exp\{i(kx + \ell y - \omega t)\}$ leads to

$$\omega_0 = 0 \quad \text{or} \quad \omega_{IG}^2 = f^2 + gHk^2 \text{ (IG waves), where } \kappa^2 = k^2 + \ell^2$$

• Depending on $\kappa$, the IG waves may be dominated either by rotation or by gravity (surface waves)

• Note that $|\omega_{IG}| > f$ (low-frequency cut-off because of rotation)
• Introduce a Doppler shift $U\kappa$ to the frequency to take account of the nonlinear advection term in the dynamics. Then

$$\frac{\omega_0}{\omega_{IG}} \approx \frac{U\kappa}{U\kappa + \sqrt{f^2 + gH\kappa^2}} \ll 1 \iff \varepsilon = \frac{U\kappa}{\sqrt{f^2 + gH\kappa^2}} \ll 1$$

• Under these conditions we have a separation in timescales, and can distinguish between fast and slow dynamics

• Define $Ro = \frac{U\kappa}{f}$ (Rossby no.) and $Fr = \frac{U}{\sqrt{gH}}$ (Froude no.) (note analogue of $Fr$ for continuously stratified flow). Then

$$\varepsilon = \frac{Ro \cdot Fr}{\sqrt{Ro^2 + Fr^2}}$$

• Hence $\varepsilon \ll 1$ if either $Ro$ or $Fr$ are small, even if the other parameter diverges (Saujani & Shepherd 2006 JFM)
• The ratio $Fr/Ro$ determines whether rotation or gravity dominates

$$\frac{Fr}{Ro} = \frac{f}{\kappa \sqrt{gH}} = \frac{L}{L_R}; \quad L=1/\kappa, L_R = \text{Rossby deformation radius}$$

• In classical quasi-geostrophic scaling, $Ro = Fr$ and the effects of rotation and gravity are taken to be comparable

• Now non-dimensionalize the full shallow-water equations with characteristic velocity scale $U$, horizontal length scale $L_R$, the slow advective time scale $L_R/U$, and $h = H (1 + Ro \, \eta)$

• This gives the dimensionless system

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{u} + \frac{1}{Ro} (\hat{z} \times \tilde{u} + \nabla \eta) = 0, \quad \frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \tilde{u}) + \frac{1}{Ro} \nabla \cdot \tilde{u} = 0$$

• If $Ro << 1$, then to obtain slow motion we require...
\[ \hat{z} \times \vec{u} + \nabla \eta = 0, \quad \nabla \cdot \vec{u} = 0, \quad \text{i.e.} \quad u = -\frac{\partial \eta}{\partial y}, \quad v = \frac{\partial \eta}{\partial x} \quad \text{(geostrophic balance)} \]

- Equivalently, \( \psi = \eta \) and \( \chi = 0 \) where \( \vec{u} = \hat{z} \times \nabla \psi + \nabla \chi \)
- But there are actually only two independent constraints here; eliminating the \( 1/Ro \) terms in the equations gives

\[
\left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right)\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta \right) = 0 \quad \text{(QG potential vorticity equation)}
\]

- This is now a closed, first-order time-evolution equation

\[
\frac{\partial q}{\partial t} + J(\psi,q) = 0 \quad \text{where} \quad q = \nabla^2 \psi - \psi
\]

(Note requirement of additional boundary condition on the circulation along lateral sidewalls)
- The existence of a single slow equation is no accident, since we had a slow (zero-frequency) linear mode in the system
Note that (as with incompressibility or hydrostatic balance) the reduction is not exact; but an asymptotic procedure can be developed to define an approximate slow manifold, with the “fast” variables \( f \) slaved to the “slow” variables \( s \) (Warn et al. 1995 QJRMS)

\[
\frac{\partial f}{\partial t} + \frac{\Gamma f}{\varepsilon} = F(s,f;\varepsilon), \quad \frac{\partial s}{\partial t} = S(s,f;\varepsilon)
\]

\[
f = U(s;\varepsilon) \implies f = f^{(0)} + \varepsilon f^{(1)} + \ldots
\]

\( O(1) \) : \quad f^{(0)} = 0, \quad \frac{\partial s}{\partial t} = S(s,0;0)

\( O(\varepsilon) \) : \quad f^{(1)} = \Gamma^{-1} F(s,0;0), \quad \frac{\partial s}{\partial t} = S(s,f^{(1)};\varepsilon)_{O(\varepsilon)}

and so on (Bokhove & Shepherd 1996 J.Atmos.Sci. show the asymptotic nature of this series for a low-order ODE system)
• In geophysical fluid dynamics, a slow equation is always provided by the **potential vorticity**, which necessarily evolves on the advective timescale (Hoskins et al. 1985 *QJRMS*)

• In the special case of the shallow-water equations, 

\[
Q = \frac{f + \mathbf{\hat{z}} \cdot \nabla \times \mathbf{\bar{u}}}{h} \rightarrow \frac{f (1 + Ro \mathbf{\hat{z}} \cdot \nabla \times \mathbf{\bar{u}})}{H(1 + Ro \eta)} \\
\approx \frac{f}{H} \left[ 1 + Ro \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \eta \right) + \ldots \right]
\]

which apart from constants is the QG potential vorticity \( q \), and the fast variables are \( \delta = \nabla \cdot \mathbf{\bar{u}} = \nabla^2 \chi \) and \( \nabla^2 (\psi - \eta) \)

• The slaving relations are known as **balance relations**, and the slow dynamics is known as **balanced dynamics**
  – \( O(1) \) is QG balance, \( O(\varepsilon) \) is Charney-Bolin balance
Potential vorticity slaving in the shallow-water system

\[ \frac{\partial q}{\partial t} = -J(\psi, q) - \nabla \chi \cdot \nabla q, \]

\[ \frac{\partial D}{\partial t} - \frac{\Omega}{\varepsilon} = -J(\chi, \nabla^2 \psi) - \frac{1}{2} \nabla^2 |\nabla \chi|^2 + \nabla^2 J(\chi, \psi) + 2J(\psi_x, \psi_y) \]

\[ \frac{\partial \Omega}{\partial t} - \frac{\mathcal{H}D}{\varepsilon} = -J(\psi, \nabla^2 \psi) - \nabla \cdot (\nabla^2 \psi \nabla \chi) + \nabla^2 \{J(\psi, \eta) + \nabla \cdot (\eta \nabla \chi)\} \]

\[ 1 + \varepsilon q = \frac{1 + \varepsilon \nabla^2 \psi}{1 + \varepsilon \eta} \quad \Leftrightarrow \quad q = \frac{\nabla^2 \psi - \eta}{1 + \varepsilon \eta} \]

\[ D = \nabla \cdot \mathbf{v} = \nabla^2 \chi \quad \Omega = \nabla^2 \psi - \nabla^2 \eta \]

\[ \mathcal{H} \equiv \nabla^2 - 1 \quad \mathbf{v} = \mathbf{k} \times \nabla \psi + \nabla \chi. \]

Warn et al. (1995 QJRMS)
\( O(1): \quad \chi^{(0)} = 0 \quad \Omega^{(0)} = 0 \quad \nabla^2 \psi^{(0)} - \psi^{(0)} = q \)
\[
\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0
\]

\( O(\varepsilon): \quad \Omega^{(1)} = -2J(\psi_x^{(0)}, \psi_y^{(0)}) \quad \Lambda \chi^{(1)} = J(\psi^{(0)}, \nabla^2 \psi^{(0)}) \)
\[
\nabla^2 \psi^{(1)} - \eta^{(1)} = \eta^{(0)} q 
\]
\[
\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0 \quad \psi = \psi^{(0)} + \varepsilon \psi^{(1)} \quad \chi = \chi^{(0)} + \varepsilon \chi^{(1)}
\]
\[
\Lambda \chi = \varepsilon J(\gamma, \nabla^2 \gamma)
\]
\[
\Lambda = \mathcal{H} \nabla^2 \quad \gamma = \psi^{(0)} = \mathcal{H}^{-1} q
\]
\[
\Lambda \psi = \nabla^2 q + \varepsilon \{ \nabla^2 (\gamma q) + 2J(\gamma_x, \gamma_y) \}
\]

Warn et al. (1995 QJRMS)
\[ O(\varepsilon^2): \quad \frac{\partial q}{\partial t} + v \cdot \nabla q = 0 \]

\[ \Lambda \psi = \nabla^2 q + \varepsilon R(\gamma) + \varepsilon^2 P(\gamma) \quad \Lambda \chi = \varepsilon G(\gamma) + \varepsilon^2 C(\gamma) \]

\[ G(\gamma) = J(\gamma, \nabla^2 \gamma) \]

\[ R(\gamma) = \nabla^2 (\gamma q) + 2J(\gamma_x, \gamma_y) \]

\[ C(\gamma) = 2J(\partial_x \{ \mathcal{H}^{-1}J(\gamma, q) \}, \gamma_y) + 2J(\gamma_x, \partial_y \{ \mathcal{H}^{-1}J(\gamma, q) \}) + J(\Lambda^{-1}R(\gamma), \nabla^2 \gamma) + \]
\[ \quad + J(\gamma, \nabla^2 \Lambda^{-1}R(\gamma)) + \nabla \cdot \{ \nabla^2 \gamma \nabla (\Lambda^{-1}G(\gamma)) \} - \]
\[ \quad - \nabla^2 [J(\gamma, \mathcal{H}^{-1}R(\gamma) - \gamma q) + J(\Lambda^{-1}R(\gamma), \gamma) + \nabla \cdot \{ \gamma \nabla (\Lambda^{-1}G(\gamma)) \} ] \]

\[ P(\gamma) = \nabla^2 \{ (\mathcal{H}^{-1}R(\gamma) - \gamma q)q \} + \mathcal{H}^{-1}J(\mathcal{H}^{-1}J(\gamma, q), \nabla^2 \gamma) + \]
\[ \quad + \mathcal{H}^{-1}J(\gamma, \nabla^2 \mathcal{H}^{-1}J(\gamma, q)) - J(\Lambda^{-1}G(\gamma), \nabla^2 \gamma) + \nabla^2 J(\Lambda^{-1}G(\gamma), \gamma) + \]
\[ \quad + 2J(\partial_x (\Lambda^{-1}R(\gamma)), \gamma_y) + 2J(\gamma_x, \partial_y (\Lambda^{-1}R(\gamma))), \]

Warn et al. (1995 QJRMS)
On the Existence of a Slow Manifold

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On the Nonexistence of a Slow Manifold

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The Slow Manifold—What Is It?

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• The Lorenz (1986 JAS) 5-component model
• Turns out to be a Hamiltonian system, with invariants
  \[ H' = \frac{1}{2} (x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2) \quad C = \frac{1}{2}(x_1^2 + x_2^2) \]
• Integrable for \( b=0 \) (pendulum coupled to harmonic oscillator)

\[
\begin{align*}
\frac{dx_1}{dt} &= -x_2 x_3 + bx_2 x_5, \\
\frac{dx_2}{dt} &= x_1 x_3 - bx_1 x_5, \\
\frac{dx_3}{dt} &= -x_1 x_2, \\
\frac{dx_4}{dt} &= -\frac{x_5}{\epsilon}, \\
\frac{dx_5}{dt} &= \frac{x_4}{\epsilon} + bx_1 x_2.
\end{align*}
\]

\[
\begin{align*}
x_1 &= \sqrt{2C} \cos \phi, \quad x_2 = \sqrt{2C} \sin \phi \\
\frac{d\phi}{dt} &= x_3 - bx_5, \\
\frac{dx_3}{dt} &= -C \sin 2\phi, \\
\frac{dx_4}{dt} &= -\frac{x_5}{\epsilon}, \\
\frac{dx_5}{dt} &= \frac{x_4}{\epsilon} + bC \sin 2\phi.
\end{align*}
\]
• In the integrable case, solutions live on a two-torus, and generally are quasi-periodic (not periodic)

Bokhove & Shepherd (1996 JAS)
- KAM theorem applies: most invariant tori preserved for sufficiently small perturbations $b$
- Poincaré sections: $b$ increasing from zero (left)

Bokhove & Shepherd (1996 JAS)
- An exact slow manifold can be defined geometrically
- Slaving approximation is only asymptotic
- Lorenz (1986) found the slow solutions as periodic solutions

Bokhove & Shepherd (1996 JAS)
• Lorenz (1986) system is pathological because slow dynamics is integrable: unambiguous separation of fast and slow dynamics
• Making $C$ periodic in time makes the slow dynamics chaotic
• Fast action is still bounded for finite times by adiabatic invariance (Nekhoroshev 1971, Neishtadt 1984)
Power spectra of potential vorticity and divergence (minimum frequency is 22)

Power spectra of divergence for $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^3)$ initialization (slaving)

Power spectrum of unslaved divergence in the last case (minimum frequency is 10)

Wirosoetisno, Shepherd & Temam (2002 *J.Atmos.Sci.*)

A yet more complex model
Instantaneous height/velocity and divergence fields
The same, but derived solely from potential vorticity
The instantaneous potential vorticity field