Applications of Hamiltonian theory to GFD

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1 Generalized Hamiltonian dynamics

1.1 Introduction

Virtually every model used in Geophysical Fluid Dynamics (GFD) is, in its conservative form, Hamiltonian. This is not too surprising since the fundamental equations from which every model is derived are themselves Hamiltonian: namely the three-dimensional Euler equations for compressible, stratified flow (Morrison & Greene 1980; Morrison 1982).

The Hamiltonian formulation of dynamics is relevant to the description of many different phenomena. In the field of theoretical physics, it provides a general foundation for quantum mechanics, quantum field theory, statistical mechanics, relativity, optics and celestial mechanics. Hamiltonian structure constitutes a unifying framework, wherein symmetry properties are readily apparent which may be connected to conservation laws by Noether's theorem. One therefore expects some of the same advantages to hold in GFD.

In these lectures we will consider particularly the application of Hamiltonian structure to problems involving disturbances to basic states. As we shall see, such diverse topics as available potential energy, wave action, and most of the well-known hydrodynamical stability theorems (static stability, symmetric stability, centrifugal stability, and the Rayleigh-Kuo and Charney-Stern theorems) may all be understood — and in some cases significantly generalized — within the Hamiltonian framework.

It is sometimes objected that Hamiltonian structure is irrelevant to GFD because real fluids are viscous. Against this, we note simply that many phenomena in GFD are essentially conservative (inviscid, adiabatic) since they occur at high Reynolds numbers, $Re \gg 1$. For example, in the free atmosphere $Re \sim 10^{15}$. Thus many GFD phenomena (instabilities, wave propagation, and wave, mean-flow interaction) are traditionally studied within the framework of a conservative model. Even if non-conservative effects arise, these may often be understood as localized effects on otherwise conserved quantities: examples include fronts, shocks, and gravity-wave drag (cf. Benjamin & Bowman 1987).

Moreover, many of the most interesting phenomena in GFD arise from the *nonlinear* (usually advective) terms in the relevant equations. Examples include wave, mean-flow interaction, energy budgets and conversions, and spectral transfers in turbulent flow. These nonlinear terms are conservative, and are therefore part of the Hamiltonian structure of the problem. It follows that the nonlinear interactions are constrained by preservation of invariant quantities (e.g. energy, enstrophy) which are connected to the underlying Hamiltonian structure of the model: one cannot deduce the correct spectral transfers in a problem unless one imposes the correct invariants on the nonlinear dynamics.

Hamiltonian structure also provides a natural framework within which to derive approximate models. It is well known that in making approximations one should attempt to maintain fundamental conserved quantities. A good example of this is provided by the hydrostatic primitive equations on the sphere (Lorenz 1967), where energy and angular momentum conservation are lost under the hydrostatic approximation, and certain manipulations must be made to the equations in order to restore them. Rather than such trial-and-error methods, it is preferable to ensure maintenance of invariance properties by making the approximations within a Hamiltonian framework (Salmon 1983, 1985, 1988*a*).

The approach followed in these lectures is to use the Hamiltonian structure of GFD in a very practical way. In particular, there is no need to use the Poisson bracket itself, or even to know it, if one knows the invariants. One needs merely to know that the bracket is there! All the manipulations required here can be expressed in terms of standard variational calculus: one has merely to vary all dependent variables, integrate by parts, and check the boundary conditions. Finally, everything derived from Hamiltonian theory may always be verified afterwards by direct use of the equations of motion.

1.2 Dynamics

We consider the generalized Hamiltonian dynamical system

$$\frac{\partial \boldsymbol{u}}{\partial t} = J\frac{\delta \mathcal{H}}{\delta \boldsymbol{u}}, \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ are the dynamical fields, $\mathcal{H}$ is the Hamiltonian, and J is a skew-symmetric operator (called the *cosymplectic form*) having the required algebraic properties (see Morrison's lectures). The equivalent formulation in terms of Poisson brackets is

$$\frac{d\mathcal{F}}{dt} = [\mathcal{F},\mathcal{H}], \tag{2}$$

where $\mathcal{F}[\boldsymbol{u}]$ is an admissible functional. The Poisson bracket is defined by

$$[\mathcal{F},\mathcal{G}] = \left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{u}}, J\frac{\delta \mathcal{G}}{\delta \boldsymbol{u}} \right\rangle \tag{3}$$

(the angle brackets denoting an appropriate inner product), and the bracket satisfies properties analogous to those of J. Typically

$$\left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{u}}, J\frac{\delta \mathcal{G}}{\delta \boldsymbol{u}} \right\rangle = \int \delta\boldsymbol{x} \sum_{i,j} \frac{\delta \mathcal{F}}{\delta u_i} J_{ij} \frac{\delta \mathcal{G}}{\delta u_j}; \tag{4}$$

i.e. the inner product is the spatial integral of the dot product of the two vectors. Further discussion of the forms (1) and (2) as applied to fluid dynamics may be found in Morrison (1982), Benjamin (1984), Salmon (1988*b*), and Shepherd (1990, 1992*a*).

Let us verify the equivalence of the above two formulations, (1) and (2). Assuming first that (1) holds, we note from (3) that

$$[\mathcal{F},\mathcal{H}] = \left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{u}}, J\frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} \right\rangle = \left\langle \frac{\delta \mathcal{F}}{\delta \boldsymbol{u}}, \frac{\partial \boldsymbol{u}}{\partial t} \right\rangle = \frac{d\mathcal{F}}{dt} \tag{5}$$

(the last step invoking the chain rule for functionals), and hence (2) is verified. Now assuming that (2) holds, let us take

$$\mathcal{F}[\boldsymbol{u}] = u_i(\boldsymbol{x}_o) = \int \delta(\boldsymbol{x} - \boldsymbol{x}_o)\, u_i(\boldsymbol{x})\, d\boldsymbol{x} \tag{6}$$

for some i and some $\boldsymbol{x}_o$, where $\delta(\boldsymbol{x} - \boldsymbol{x}_o)$ is the Dirac delta-function; thus

$$\delta\mathcal{F} = \int \delta(\boldsymbol{x} - \boldsymbol{x}_o)\delta u_i(\boldsymbol{x})\, d\boldsymbol{x}, \qquad \frac{\delta\mathcal{F}}{\delta u_j} = \delta(\boldsymbol{x} - \boldsymbol{x}_o)\, \delta_{ij}, \tag{7}$$

where δ_{ij} is the Kronecker delta. Then using (2), (6) and (7), we have

$$\frac{\partial u_i}{\partial t}(\boldsymbol{x}_o) = \frac{d\mathcal{F}}{dt} = \left\langle \frac{\delta\mathcal{F}}{\delta \boldsymbol{u}}, J\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}} \right\rangle = \left\langle \delta(\boldsymbol{x} - \boldsymbol{x}_o)\delta_{ij}, \left(J\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}}\right)_j \right\rangle = \left(J\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}}\right)_i(\boldsymbol{x}_o). \tag{8}$$

Thus (1) is verified, component by component.

1.3 Steady states and conditional extrema

Let $\boldsymbol{u} = \boldsymbol{U}$ be a steady solution of the dynamics (1). If J is invertible, then

$$J\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} = \frac{\partial \boldsymbol{U}}{\partial t} = 0 \tag{9}$$

leads to

$$\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} = 0. \tag{10}$$

Hence steady solutions are extrema of $\mathcal{H}$.

But suppose now that the dynamics of the system is non-canonical, in the sense that J is non-invertible (cf. Morrison's lectures). Then (9) does *not* imply (10). However, Casimirs $\mathcal{C}$ may be defined such that

$$J\frac{\delta\mathcal{C}}{\delta \boldsymbol{u}} = 0 \qquad \text{(equivalently, } [\mathcal{C}, \mathcal{F}] = 0\ \forall\, \mathcal{F}\text{)}, \tag{11}$$

and the set of all vectors $\delta\mathcal{C}/\delta\boldsymbol{u}$ spans the kernel of J. At $\boldsymbol{u} = \boldsymbol{U}$, therefore, $\delta\mathcal{H}/\delta\boldsymbol{u}$ is locally parallel to $\delta\mathcal{C}/\delta\boldsymbol{u}$ for some $\mathcal{C}$ (a different $\mathcal{C}$ for each choice of $\boldsymbol{U}$); equivalently, there generically exists a Casimir $\mathcal{C}$ such that

$$\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} = -\frac{\delta\mathcal{C}}{\delta \boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}}. \tag{12}$$

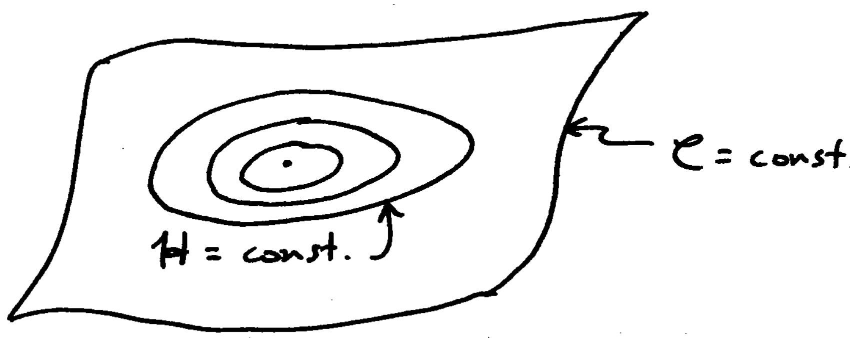
One must be careful here with classes of admissible variations; this point will come up again when we consider nonlinear stability. Note that Casimirs are always invariants of the motion, since

$$\frac{d\mathcal{C}}{dt} = [\mathcal{C}, \mathcal{H}] = \left\langle \frac{\delta\mathcal{C}}{\delta \boldsymbol{u}}, J\frac{\delta\mathcal{H}}{\delta \boldsymbol{u}} \right\rangle = -\left\langle J\frac{\delta\mathcal{C}}{\delta \boldsymbol{u}}, \frac{\delta\mathcal{H}}{\delta \boldsymbol{u}} \right\rangle = 0. \tag{13}$$

From (12), we have that

$$\frac{\delta}{\delta \boldsymbol{u}}(\mathcal{H}+\mathcal{C})\Big|_{\boldsymbol{u}=\boldsymbol{U}} = 0. \tag{14}$$

This statement has two interpretations: (i) $\boldsymbol{U}$ is an extremum of the invariant $\mathcal{H}+\mathcal{C}$; and (ii) $\boldsymbol{U}$ is a conditional extremum of $\mathcal{H}$, subject to the constraint $\mathcal{C} = \text{const.}$ (as with Lagrange multipliers). An example of an elliptic fixed point, representing a maximum or minimum, is sketched below; the curves are lines of constant $\mathcal{H}$, and the constraint surface is the "symplectic leaf" $\mathcal{C} = \text{const.}$



1.4 Example: barotropic vorticity equation

This model is discussed in Morrison's lectures, but it is useful to consider it in the present context. The discussion will also illustrate some of the complications that are introduced by boundaries. The governing equation is the (2-D) vorticity equation

$$\frac{\partial \omega}{\partial t} + \partial(\psi, \omega) = 0\,, \tag{15}$$

where ψ is the stream function, the velocity is given by $\boldsymbol{v} = \hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi$, $\omega = \nabla^2\psi$ is the vorticity, and $\partial(a,b) \equiv a_x b_y - a_y b_x$ is the two-dimensional Jacobian operator. With this choice of ω, the system is identical to the 2-D Euler equations. We consider a closed, multiply-connected domain D with N connected boundaries ∂D_i ($i = 1, \ldots, N$) on which $\boldsymbol{v}\cdot\hat{\mathbf{n}} = 0$ (or $\partial\psi/\partial s = 0$, where s is arclength along ∂D_i), where $\hat{\mathbf{n}}$ is the unit outward normal vector.

This system is Hamiltonian with

$$\mathcal{H} = \iint_D \frac{1}{2}|\boldsymbol{\nabla}\psi|^2\, dxdy. \tag{16}$$

The first variation of $\mathcal{H}$ is given by

$$\begin{aligned} \delta\mathcal{H} &= \iint_D \boldsymbol{\nabla}\psi\cdot\delta\boldsymbol{\nabla}\psi\, dxdy \\ &= \iint_D \left[\boldsymbol{\nabla}\cdot(\psi\delta\boldsymbol{\nabla}\psi) - \psi\,\delta\nabla^2\psi\right]\, dxdy \\ &= \sum_i \psi\,\delta \oint_{\partial D_i} \boldsymbol{\nabla}\psi\cdot\hat{\mathbf{n}}\, ds - \iint_D \psi\delta\omega\, dxdy, \end{aligned} \tag{17}$$

where the last step follows from the fact that ψ is constant on the boundaries. This means that one cannot write $\delta\mathcal{H} = \langle(\delta\mathcal{H}/\delta\omega), \delta\omega\rangle$ alone. Stated otherwise, ω is not enough to determine the dynamics; we need boundary terms as well, as follows.

Defining $\gamma_i \equiv \oint_{\partial D_i} \nabla\psi \cdot \hat{\mathbf{n}}\, ds$ to be the circulation on each connected piece ∂D_i of ∂D, recall that $d\gamma_i/dt = 0$. (This is the usual boundary condition on the tangential velocity; it follows from consideration of the momentum equations underlying (15).) The boundary circulations can therefore be considered dynamical variables, and one may rewrite $\delta\mathcal{H}$ in terms of $\delta\gamma_i$ in addition to $\delta\omega$: from (17),

$$\delta\mathcal{H} = \sum_i \psi\, \delta\gamma_i - \iint_D \psi \delta\omega\, dxdy\,, \tag{18}$$

which implies

$$\frac{\delta\mathcal{H}}{\delta\omega} = -\psi, \qquad \frac{\delta\mathcal{H}}{\delta\gamma_i} = \psi\Big|_{\partial D_i}. \tag{19}$$

Note that in the first equation of (19), one cannot think in terms of partial derivatives: in particular, $\partial|\boldsymbol{v}|^2/\partial\omega$ makes no sense. Instead, it is clear that variational derivatives are required.

Relative to ω alone, the γ_i's extend the phase space in the following way: there are now $N+1$ dynamical variables $\boldsymbol{u} = (\omega, \gamma_1, \ldots, \gamma_N)^T$, and the cosymplectic form J is an $(N+1)\times(N+1)$ matrix operator:

$$J = \begin{pmatrix} -\partial(\omega,\cdot) & 0 & \ldots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \end{pmatrix}. \tag{20}$$

Substituting (19) and (20) into (1) yields, as expected, the equations of motion

$$\Big(\frac{\partial\omega}{\partial t}, \frac{d\gamma_1}{dt}, \ldots, \frac{d\gamma_N}{dt}\Big)^T = \frac{\partial\boldsymbol{u}}{\partial t} = J\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} = \big(-\partial(\omega, -\psi), 0, \ldots, 0\big)^T. \tag{21}$$

Having seen that arbitrary disturbances can be incorporated into the Hamiltonian description, let us now, for simplicity, restrict our attention to circulation-preserving disturbances: namely those with $\delta\gamma_i = 0$ for all i. (If this condition holds at one time, it will hold at all subsequent times.) For this special case, ω is the sole dynamical variable and $J = -\partial(\omega,\cdot)$. Let us find the Casimirs. Solving (11) in this case, we obtain

$$\partial(\omega, \frac{\delta\mathcal{C}}{\delta\omega}) = 0\,; \tag{22}$$

in other words, lines of constant ω and constant $\delta\mathcal{C}/\delta\omega$ coincide. Locally, at least, this means that $\delta\mathcal{C}/\delta\omega = f(\omega)$ for some function f. Such a function may not be defined over the entire domain D, however. A sub-class of these Casimirs which is useful for applications (see the later sections on stability) consists of those for which the functional relation is global: these may be written as

$$\mathcal{C}[\omega] = \iint_D C(\omega)\, dxdy \tag{23}$$

for some function C. Since Casimirs are always invariants of the motion, this demonstrates that

$$\frac{d}{dt}\iint_D C(\omega)\, dxdy = 0 \tag{24}$$

for *any* function $C(\omega)$. The set of conservation laws described by (24) reflects the fact that ω is a Lagrangian or material invariant of the dynamics (15), given that the flow is non-divergent. Since the dynamical evolution takes place on the symplectic leaf $\mathcal{C}$ =const., where the constraint refers to *all* Casimirs simultaneously, we see that the Casimirs provide a severe restriction on dynamically possible behaviour. This is intuitively obvious for piecewise-constant vorticity profiles. The calculation also demonstrates that there is nothing esoteric about Casimirs: they have real physical meaning.

We should be able to show that steady solutions of (15) are conditional extrema of $\mathcal{H}$, subject to the constraint that the variations preserve $\mathcal{C}$. First consider the extremal condition (12), which takes the form $\psi = C'(\omega)$ in this case for $\mathcal{C}$ given by (23). If, therefore, (12) holds, it follows that

$$\omega_t = -\partial(\psi,\omega) = -\partial(C'(\omega),\omega) = 0, \tag{25}$$

and the flow is steady. One may also build in the constraint imposed by conservation of $\mathcal{C}$ directly on the variations. To do this, set $\delta\omega = \partial(\varphi,\omega)$ for some arbitrary φ which is constant on the boundaries. Such variations $\delta\omega$ are clearly just non-divergent (area-preserving) rearrangements of the vorticity field ω, for which

$$\delta\mathcal{C} = \iint_D \frac{\delta\mathcal{C}}{\delta\omega}\,\delta\omega\,dxdy = \iint_D C'(\omega)\partial(\varphi,\omega)\,dxdy = \iint_D \partial(\varphi, C(\omega))\,dxdy = 0. \tag{26}$$

For steady states with $\partial(\psi,\omega) = 0$, the variation of $\mathcal{H}$ is then

$$\delta\mathcal{H} = \iint_D \frac{\delta\mathcal{H}}{\delta\omega}\,\delta\omega\,dxdy = -\iint_D \psi\partial(\varphi,\omega)\,dxdy = \iint_D \varphi\partial(\psi,\omega)\,dxdy = 0 \tag{27}$$

(using the fact that both ψ and φ are constant on the boundary); hence steady solutions of (15) are seen to be unconditional extrema of $\mathcal{H}$ for vorticity-preserving variations, as expected on general grounds.

The variations $\delta\omega = \partial(\varphi,\omega)$ considered above may be written in the form $\delta\omega = J\varphi$, which suggests the general form $\delta\boldsymbol{u} = J\varphi$ for a vector φ. Evidently such variations are guaranteed to be Casimir-preserving, since

$$\delta\mathcal{C} = \left\langle \frac{\delta\mathcal{C}}{\delta\boldsymbol{u}}, \delta\boldsymbol{u} \right\rangle = \left\langle \frac{\delta\mathcal{C}}{\delta\boldsymbol{u}}, J\varphi \right\rangle = -\left\langle J\frac{\delta\mathcal{C}}{\delta\boldsymbol{u}}, \varphi \right\rangle = 0. \tag{28}$$

The reader is referred to Morrison's notes for a more detailed description of such variations, which he refers to as being "dynamically accessible".

1.5 Symmetries and conservation laws

As in textbook classical mechanics (e.g. Goldstein 1980), for any functional $\mathcal{F}$ we can define a one-parameter family of infinitesimal variations $\delta_{\mathcal{F}}\boldsymbol{u}$ induced by $\mathcal{F}$ by

$$\delta_{\mathcal{F}}\boldsymbol{u} = \epsilon J \frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}, \tag{29}$$

where ϵ is the infinitesimal parameter. The change in another functional $\mathcal{G}$ induced by this variation is

$$\Delta_{\mathcal{F}}\mathcal{G} \equiv \mathcal{G}[\boldsymbol{u} + \delta_{\mathcal{F}}\boldsymbol{u}] - \mathcal{G}[\boldsymbol{u}] = \left\langle \frac{\delta\mathcal{G}}{\delta\boldsymbol{u}}, \delta_{\mathcal{F}}\boldsymbol{u} \right\rangle + \mathcal{O}((\delta_{\mathcal{F}}\boldsymbol{u})^2) = \epsilon[\mathcal{G},\mathcal{F}] + \mathcal{O}(\epsilon^2), \tag{30}$$

where the second step follows from the definition of the functional derivative, and the third step from the definition of the bracket together with (29). This proves

Noether's Theorem: The Hamiltonian is invariant under infinitesimal variations generated by a functional $\mathcal{F}$, in the sense that $\Delta_{\mathcal{F}}\mathcal{H} = 0$, if and only if $\mathcal{F}$ is a constant of the motion.

Therefore, given a symmetry of the Hamiltonian (a variation $\delta \boldsymbol{u}$ under which the Hamiltonian is invariant), one can attempt to solve (29) to find the corresponding invariant (modulo a Casimir). Equally, given a known invariant $\mathcal{F}$, one can use (29) to determine the corresponding symmetry.

Exercise: Cyclic coordinates in a finite-dimensional canonical system. If $\mathcal{H}$ is invariant under translations in q_i (i.e. $\partial\mathcal{H}/\partial q_i = 0$ for some i), use (29) to show that the corresponding p_i is a constant of the motion.

As is well known, the KdV equation possesses more than one (non-trivially related) Hamiltonian representation. Consider two representations with cosymplectic forms J_1 and J_2. Suppose that δu_1 is a symmetry of the system; using J_1 with (29) then defines an invariant I_1. But knowing I_1, (29) may now be used with J_2 to find a new symmetry, δu_2. Then substituting δu_2 back into (29) with J_1 produces a new invariant I_2, and so on. This procedure will continue indefinitely as long as we keep generating new invariants; in the case of the KdV equation this turns out to be true, and leads to exact integrability. See Olver (1986) for a more thorough, and highly readable, discussion of this topic.

Returning to the relation (29), we see that Casimirs correspond to *invisible* symmetries since

$$\delta_{\mathcal{C}}\boldsymbol{u} = \epsilon J \frac{\delta \mathcal{C}}{\delta \boldsymbol{u}} = 0 : \tag{31}$$

Casimirs induce no change whatsoever in the dynamical variables.

Let us now consider some examples of symmetries and conservation laws. First suppose that the Hamiltonian $\mathcal{H}$ is invariant under translation in time. We can set $\delta_{\mathcal{F}}\boldsymbol{u} = -\epsilon(\partial \boldsymbol{u}/\partial t)$ as the variation in $\boldsymbol{u}$ induced by a shift in time, $\epsilon = \delta t$. (The minus sign is indeed correct: think about it!) To find the corresponding invariant $\mathcal{F}$ we must therefore solve $-(\partial \boldsymbol{u}/\partial t) = J(\delta\mathcal{F}/\delta\boldsymbol{u})$, which implies $\mathcal{F} = -\mathcal{H}$ (to within a Casimir). This shows that $\mathcal{H}$ is the invariant corresponding to time-translation invariance. (This statement is not trivial. In particular, recall the relation $d\mathcal{H}/dt = \partial\mathcal{H}/\partial t$ in classical mechanics; the former corresponds to a conservation law, the latter to a symmetry-invariance.)

As another example, suppose that the Hamiltonian $\mathcal{H}$ is invariant under translation in space: x_j, say, for some j. We can set $\delta_{\mathcal{F}}\boldsymbol{u} = -\epsilon(\partial\boldsymbol{u}/\partial x_j)$, and to find the corresponding invariant we must solve $-(\partial\boldsymbol{u}/\partial x_j) = J(\delta\mathcal{F}/\delta\boldsymbol{u})$. In the case of the barotropic vorticity equation, for example, with $j = 1$ this becomes

$$\begin{aligned} \frac{\partial\omega}{\partial x} &= \partial\Big(\omega, \frac{\delta\mathcal{F}}{\delta\omega}\Big) \qquad \Longrightarrow \qquad \frac{\delta\mathcal{F}}{\delta\omega} = y \\ \Longrightarrow \qquad \mathcal{F} &= \iint_D y\omega\, dxdy = \iint_D y\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\, dxdy = \iint_D u\, dxdy \end{aligned} \tag{32}$$

(to within a Casimir). Therefore the invariant corresponding to x-translation invariance of the dynamics is seen to be the zonal momentum, as expected.

For $j = 2$, similar considerations lead to

$$\begin{aligned} \frac{\partial \omega}{\partial y} &= \partial(\omega, \frac{\delta \mathcal{F}}{\delta \omega}) \qquad \Longrightarrow \qquad \frac{\delta \mathcal{F}}{\delta \omega} = -x \\ \Longrightarrow \qquad \mathcal{F} &= -\iint_D x\omega \, dxdy = \iint_D x\Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big)\, dxdy = \iint_D v \, dxdy \end{aligned} \tag{33}$$

(also to within a Casimir). Therefore the invariant corresponding to y-translation invariance of the dynamics is seen to be the meridional momentum.

By Noether's theorem, the same construction is guaranteed to work for *any* continuous symmetry. Let us show it for a rotation. We take the variation to be $\delta r = 0$, $\delta\theta = \epsilon$, where r and θ are polar coordinates defined by $x = r\cos\theta$ and $y = r\sin\theta$. The corresponding variations in x and y are given by

$$\delta x = -r\sin\theta\,\delta\theta = -y\epsilon, \qquad \delta y = r\cos\theta\,\delta\theta = x\epsilon. \tag{34}$$

It follows that the variation in the dynamical variable ω is

$$\delta\omega = -\frac{\partial \omega}{\partial x}\delta x - \frac{\partial \omega}{\partial y}\delta y = \Big(y\frac{\partial \omega}{\partial x} - x\frac{\partial \omega}{\partial y}\Big)\epsilon. \tag{35}$$

Then to determine the invariant corresponding to this symmetry we must solve (29), which takes the form

$$\begin{aligned} x\frac{\partial \omega}{\partial y} - y\frac{\partial \omega}{\partial x} &= \partial(\omega, \frac{\delta \mathcal{F}}{\delta \omega}) \qquad \Longrightarrow \qquad \frac{\delta \mathcal{F}}{\delta \omega} = -\frac{1}{2}(x^2 + y^2) = -\frac{r^2}{2} \\ \Longrightarrow \qquad \mathcal{F} &= -\iint_D \frac{r^2}{2}\omega\, dxdy = \iint_D \hat{\mathbf{z}} \cdot (\boldsymbol{r} \times \boldsymbol{v}) dxdy \end{aligned} \tag{36}$$

(to within a Casimir). The last computation is obtained after integrating by parts. As expected, we obtain the angular momentum.

1.6 Steadily-translating solutions

Suppose there exists a solution to the system (1) translating steadily in x at a speed c, i.e. $\boldsymbol{u}(x, y, z, t) = \boldsymbol{U}(x - ct, y, z)$. Then clearly

$$\frac{\partial \boldsymbol{U}}{\partial t} = -c\frac{\partial \boldsymbol{U}}{\partial x}. \tag{37}$$

The fact that the solution is translating in x implies that there is a symmetry in x; if $\mathcal{M}$ is the invariant corresponding to this symmetry, then by (29)

$$-\frac{\partial \boldsymbol{U}}{\partial x} = J\frac{\delta \mathcal{M}}{\delta \boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}}. \tag{38}$$

On the other hand, we have

$$\frac{\partial \boldsymbol{U}}{\partial t} = J\frac{\delta \mathcal{H}}{\delta \boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}}. \tag{39}$$

It follows from (37), (38), and (39) that

$$J\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} = cJ\frac{\delta\mathcal{M}}{\delta\boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} \implies J\frac{\delta(\mathcal{H}-c\mathcal{M})}{\delta\boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} = 0$$

$$\implies \frac{\delta(\mathcal{H}-c\mathcal{M}+\mathcal{C})}{\delta\boldsymbol{u}}\bigg|_{\boldsymbol{u}=\boldsymbol{U}} = 0 \tag{40}$$

for some Casimir $\mathcal{C}$. Thus $\boldsymbol{U}$ is seen to be a conditional (or constrained) extremum of the invariant $\mathcal{H} - c\mathcal{M}$. We note that (40) provides a variational principle for travelling-wave solutions (cf. Benjamin 1984).

2 Hamiltonian structure of quasi-geostrophic flow

In order to illustrate the general theory of the previous section, we describe in some detail the Hamiltonian structure of what is probably the most widely-used model in theoretical geophysical fluid dynamics: quasi-geostrophic flow. Two specific such models are considered: the two-layer model in a periodic zonal β-plane channel, and continuously stratified flow over topography.

2.1 The two-layer model

The governing equations may be written (e.g. Pedlosky 1987) as

$$\frac{\partial q_i}{\partial t} + \boldsymbol{v}_i \cdot \boldsymbol{\nabla} q_i = 0 \quad [i = 1,2], \tag{41}$$

where the velocity in each layer is given by $\boldsymbol{v}_i = \hat{\mathbf{z}} \times \boldsymbol{\nabla}\psi_i$, and the potential vorticity by

$$q_i = \nabla^2\psi_i + (-1)^i F_i(\psi_1 - \psi_2) + f + \beta y \quad [i = 1,2]. \tag{42}$$

The parameter F_i is a measure of the stratification; if the layer depths are denoted D_i, then we have the geometric constraint $D_1F_1 = D_2F_2$. All fields are assumed to be periodic in x. The boundary conditions at the channel walls $y = 0, 1$ are the usual ones of no normal flow,

$$\frac{\partial\psi_i}{\partial x} = 0 \qquad \text{at } y = 0, 1 \quad [i = 1,2]; \tag{43}$$

and conservation of circulation,

$$\frac{d}{dt}\int\frac{\partial\psi_i}{\partial y}\,dx\bigg|_{y=0} \equiv -\frac{d}{dt}\gamma_i^0 = 0, \qquad \frac{d}{dt}\int\frac{\partial\psi_i}{\partial y}\,dx\bigg|_{y=1} \equiv \frac{d}{dt}\gamma_i^1 = 0 \qquad [i = 1,2]. \tag{44}$$

The dynamical variables are q_1, q_2, γ_1^0, γ_1^1, γ_2^0, and γ_2^1. We can write the Hamiltonian as

$$\mathcal{H} = \iint_D \frac{1}{2}\Big\{D_1|\boldsymbol{\nabla}\psi_1|^2 + D_2|\boldsymbol{\nabla}\psi_2|^2 + D_1F_1(\psi_1-\psi_2)^2\Big\}\,dxdy, \tag{45}$$

in which case

$$\begin{aligned}\delta\mathcal{H} &= \iint_D \left\{D_1\nabla\psi_1\cdot\nabla\delta\psi_1 + D_2\nabla\psi_2\cdot\nabla\delta\psi_2 + D_1F_1(\psi_1-\psi_2)\delta(\psi_1-\psi_2)\right\} dxdy \\ &= \iint_D \left\{D_1\nabla\cdot(\psi_1\nabla\delta\psi_1) - D_1\psi_1\delta\nabla^2\psi_1 + D_2\nabla\cdot(\psi_2\nabla\delta\psi_2) - D_2\psi_2\delta\nabla^2\psi_2 \right.\\ &\qquad \left. + D_1F_1\,\psi_1\delta(\psi_1-\psi_2) - D_2F_2\,\psi_2\delta(\psi_1-\psi_2)\right\} dxdy. \end{aligned} \tag{46}$$

To obtain the last line, the relation $D_1F_1 = D_2F_2$ has been used. This gives

$$\begin{aligned}\delta\mathcal{H} &= D_1\psi_1\Big|_{y=1}\delta\gamma_1^1 + D_1\psi_1\Big|_{y=0}\delta\gamma_1^0 + D_2\psi_2\Big|_{y=1}\delta\gamma_2^1 + D_2\psi_2\Big|_{y=0}\delta\gamma_2^0 \\ &\quad - \iint_D \left\{D_1\psi_1\,\delta[\nabla^2\psi_1 - F_1(\psi_1-\psi_2)] + D_2\psi_2\,\delta[\nabla^2\psi_2 + F_2(\psi_1-\psi_2)]\right\} dxdy, \end{aligned} \tag{47}$$

from which we may infer

$$\frac{\delta\mathcal{H}}{\delta q_i} = -D_i\psi_i \qquad \text{and} \qquad \frac{\delta\mathcal{H}}{\delta\gamma_i^{0,1}} = D_i\psi_i\Big|_{y=0,1} \quad [i=1,2]. \tag{48}$$

The functional derivatives in this system are evidently analogous to those of the barotropic system, as described in Section 1.4. Taking the dynamical variable $\boldsymbol{u}$ to be

$$\boldsymbol{u} = (q_1, q_2, \gamma_1^0, \gamma_1^1, \gamma_2^0, \gamma_2^1)^T, \tag{49}$$

the cosymplectic form J is clearly

$$J = \begin{pmatrix} -\frac{1}{D_1}\partial(q_1,\cdot) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{D_2}\partial(q_2,\cdot) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

The Casimirs are of the form

$$\mathcal{C}[q_1, q_2, \gamma_1^0, \gamma_1^1, \gamma_2^0, \gamma_2^1] = \iint_D \left\{C_1(q_1) + C_2(q_2)\right\} dxdy + \sum_{i=1,2}^{j=0,1} C_i^j\,\gamma_i^j, \tag{51}$$

where the C_i's are arbitrary functions of one argument, and the C_i^j's are arbitrary scalars. It is easy to see that

$$\frac{\delta\mathcal{C}}{\delta q_i} = C_i'(q_i), \qquad \frac{\delta\mathcal{C}}{\delta\gamma_i^j} = C_i^j, \tag{52}$$

whence the condition (11) is verified. To find the steady states, we must solve the following extremal equations: for q_i,

$$\frac{\delta\mathcal{H}}{\delta q_i} = -\frac{\delta\mathcal{C}}{\delta q_i} \qquad \Longrightarrow \qquad D_i\psi_i = C_i'(q_i), \tag{53}$$

which implies $\psi_i = \psi_i(q_i)$; and for γ_i^j,

$$\frac{\delta\mathcal{H}}{\delta\gamma_i^j} = -\frac{\delta C}{\delta\gamma_i^j} \qquad \Longrightarrow \qquad D_i\psi_i\Big|_{y=j} = -C_i^j, \tag{54}$$

which implies that ψ_i is constant along the boundaries.

To find the zonal momentum invariant $\mathcal{M}$, we must solve the equations

$$\frac{\partial q_1}{\partial x} = \frac{1}{D_1}\partial(q_1, \frac{\delta\mathcal{M}}{\delta q_1}), \qquad \frac{\partial q_2}{\partial x} = \frac{1}{D_2}\partial(q_2, \frac{\delta\mathcal{M}}{\delta q_2}) \tag{55}$$

simultaneously. Note that there is no continuous symmetry for γ_i^j. The solution (to within a Casimir) of (55) is evidently

$$\frac{\delta\mathcal{M}}{\delta q_i} = D_i y \qquad \Longrightarrow \qquad \mathcal{M} = \iint_D \big\{D_1 y q_1 + D_2 y q_2\big\}\, dxdy, \tag{56}$$

again analogous to the barotropic case. Using the definition of q_i,

$$\begin{aligned}
\mathcal{M} &= \iint_D y\Big\{D_1\big(\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2}\big) + D_2\big(\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial y^2}\big) + (D_1 + D_2)(f + \beta y)\Big\}\, dxdy \\
&= \Big[y\int\Big(D_1\frac{\partial\psi_1}{\partial y} + D_2\frac{\partial\psi_2}{\partial y}\Big)dx\Big]_{y=0}^{y=1} - \iint_D\Big\{D_1\frac{\partial\psi_1}{\partial y} + D_2\frac{\partial\psi_2}{\partial y}\Big\}\, dxdy + \text{const.} \\
&= D_1\gamma_1^1 + D_2\gamma_2^1 + \iint_D (D_1 u_1 + D_2 u_2)\, dxdy + \text{const.}
\end{aligned} \tag{57}$$

The first two terms of the above expression are Casimirs, while the spatial integral represents the zonal momentum.

2.2 Continuously stratified flow over topography

In the above sub-section we have shown how to handle the circulation terms on the side walls, so to simplify the following manipulations we now restrict our attention to the case where the circulation is held fixed when performing the variations. We again consider a periodic zonal channel, bounded top and bottom by rigid lids, with $0 \le z \le 1$. The dynamics is given by (e.g. Pedlosky 1987)

$$\frac{Dq}{Dt} \equiv \frac{\partial q}{\partial t} + \partial(\psi, q) = 0 \quad [0 < z < 1], \tag{58}$$

$$\frac{D}{Dt}(\psi_z + fSh) = 0 \quad [z = 0], \qquad \frac{D}{Dt}(\psi_z) = 0 \quad [z = 1], \tag{59}$$

where the potential vorticity q is defined by

$$q = \psi_{xx} + \psi_{yy} + \frac{1}{\rho_s}\Big(\frac{\rho_s}{S}\psi_z\Big)_z + f + \beta y. \tag{60}$$

The density $\rho_s(z)$ and stratification function $S(z) = N^2/f^2$ (where $N(z)$ is the buoyancy frequency) are both prescribed, $h(x,y)$ is the topography at the lower surface, and ψ_z is proportional to the temperature.

The dynamical boundary conditions (59) on the lids $z = 0, 1$ are necessary, and represent true degrees of freedom. This can be seen by varying $\mathcal{H}$:

$$\mathcal{H} = \iiint_D \frac{\rho_s}{2}\left\{|\nabla\psi|^2 + \frac{1}{S}\psi_z^2\right\} dxdydz \tag{61}$$

implies

$$\begin{aligned} \delta\mathcal{H} &= \iiint_D \rho_s\left\{\nabla\psi \cdot \nabla\delta\psi + \frac{1}{S}\psi_z\delta\psi_z\right\} dxdydz \\ &= \iiint_D \left\{-\rho_s\psi\delta\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\psi\delta\psi_z\right) - \psi\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\delta\psi_z\right)\right\} dxdydz \\ &= \left[\iint \frac{\rho_s}{S}\psi\delta\psi_z\, dxdy\right]_{z=0}^{z=1} - \iiint_D \rho_s\psi\delta q\, dxdydz \end{aligned} \tag{62}$$

(noting that the variations in the side-wall circulations have been taken to vanish). In particular, we cannot write $\delta\mathcal{H} = \langle(\delta\mathcal{H}/\delta q), \delta q\rangle$ unless we treat the terms in (62) involving the spatial integrals over the lids.

One option is to make the lids isentropic: $\psi_z =$ constant. Then in a completely analogous fashion to the way in which one may eliminate the circulation terms, one may restrict attention to variations with $\delta\psi_z = 0$ on $z = 0, 1$, in which case the integrals over the lids in (62) disappear. Note that this is dynamically self-consistent: from the governing equations, it follows that isentropic lids remain isentropic under the dynamics. Pursuing this option leaves us with a dynamical structure very similar to that of the barotropic system, but this is very restrictive indeed. For example, it eliminates the meridional temperature gradient at the lower surface which is so crucial in driving the atmospheric circulation.

A better option is to incorporate the terms in question into $\delta\mathcal{H}$. This can be done by introducing additional dynamical variables, just as one may introduce the side-wall circulations as dynamical variables (see previous sub-section). It is natural to define

$$\lambda_0 = \frac{\rho_s}{S}(\psi_z + fSh)\Big|_{z=0}, \qquad \lambda_1 = \frac{\rho_s}{S}\psi_z\Big|_{z=1}, \tag{63}$$

in which case (59) take the form

$$\frac{D\lambda_0}{Dt} = 0, \qquad \frac{D\lambda_1}{Dt} = 0. \tag{64}$$

Then (62) can be written

$$\delta\mathcal{H} = \iint \psi\,\delta\lambda_1\, dxdy\Big|_{z=1} - \iint \psi\,\delta\lambda_0\, dxdy\Big|_{z=0} - \iiint_D \rho_s\psi\,\delta q\, dxdydz; \tag{65}$$

the entire variation of $\mathcal{H}$ is now captured, with the functional derivatives

$$\frac{\delta\mathcal{H}}{\delta q} = -\rho_s\psi, \qquad \frac{\delta\mathcal{H}}{\delta\lambda_0} = -\psi\Big|_{z=0}, \qquad \frac{\delta\mathcal{H}}{\delta\lambda_1} = \psi\Big|_{z=1}. \tag{66}$$

Taking the dynamical variable to be $\boldsymbol{u} = (q, \lambda_0, \lambda_1)^T$, the cosymplectic form is evidently

$$J = \begin{pmatrix} -\frac{1}{\rho_s}\partial(q,\cdot) & 0 & 0 \\ 0 & -\partial(\lambda_0,\cdot) & 0 \\ 0 & 0 & \partial(\lambda_1,\cdot) \end{pmatrix}. \tag{67}$$

The Casimirs are clearly of the form

$$\mathcal{C}[q,\lambda_0,\lambda_1] = \iiint_D \rho_s C(q)\,dxdydz + \iint C_0(\lambda_0)\,dxdy\bigg|_{z=0} + \iint C_1(\lambda_1)\,dxdy\bigg|_{z=1} \tag{68}$$

for arbitrary functions C, C_0, and C_1, with

$$\frac{\delta\mathcal{C}}{\delta q} = \rho_s C'(q), \qquad \frac{\delta\mathcal{C}}{\delta\lambda_0} = C_0'(\lambda_0), \qquad \frac{\delta\mathcal{C}}{\delta\lambda_1} = C_1'(\lambda_1), \tag{69}$$

which when combined with (67) may be seen to satisfy (11).

The steady-state solutions satisfy

$$\frac{\delta\mathcal{H}}{\delta q} = -\frac{\delta\mathcal{C}}{\delta q}, \tag{70}$$

which implies $\rho_s\psi = \rho_s C'(q)$ and thus $\psi = \psi(q)$; and

$$\frac{\delta\mathcal{H}}{\delta\lambda_i} = -\frac{\delta\mathcal{C}}{\delta\lambda_i} \quad [i=0,1], \tag{71}$$

which implies $(-1)^i\psi = C_i'(\lambda_i)$ and thus $\psi = \psi(\lambda_i)$ on $z=i$.

To find the zonal momentum invariant $\mathcal{M}$, we must solve the equations

$$\frac{\partial q}{\partial x} = \frac{1}{\rho_s}\partial\Big(q,\frac{\delta\mathcal{M}}{\delta q}\Big), \qquad \frac{\partial\lambda_0}{\partial x} = \partial\Big(\lambda_0,\frac{\delta\mathcal{M}}{\delta\lambda_0}\Big), \qquad \frac{\partial\lambda_1}{\partial x} = -\partial\Big(\lambda_1,\frac{\delta\mathcal{M}}{\delta\lambda_1}\Big) \tag{72}$$

simultaneously; the solution (to within a Casimir) is

$$\mathcal{M} = \iiint_D \rho_s y q\,dxdydz + \iint y\lambda_0\,dxdy\bigg|_{z=0} - \iint y\lambda_1\,dxdy\bigg|_{z=1}. \tag{73}$$

Exercise: Show that (to within a Casimir) $\mathcal{M} = \iiint_D \rho_s u\,dxdydz$.

3 Pseudoenergy and available potential energy

3.1 Disturbances to basic states

Very often one is interested in flows that are close to some given basic state. Examples include the energetics of waves, stability and instability of basic flows, wave propagation in inhomogeneous media, and wave, mean-flow interaction. We would therefore like a Hamiltonian description of the *disturbance* problem. Ideally it should be exact, i.e. nonlinear. Two questions immediately arise: What is the correct Hamiltonian? What is the energy? The answer to these questions involves a new quantity, often referred to as the *pseudoenergy*. One of the simplest contexts in which the relevant issues arise is the familiar and classical one of available potential energy (APE), so we shall discuss it at some length. Further details may be found in Shepherd (1993*a*).

3.2 APE of internal gravity waves

Consider the energy of internal gravity waves in an incompressible, Boussinesq fluid, governed by the equations

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + f\hat{\mathbf{z}}\times\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\rho_{00}} - \frac{\rho}{\rho_{00}}g\hat{\mathbf{z}}, \tag{74}$$

$$\rho_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{75}$$

where ρ_{00} is a constant reference density. The notation is standard. The resting basic state on which the waves exist is assumed to have a horizontally uniform density $\rho = \rho_0(z)$, with stable stratification: $g(d\rho_0/dz) < 0$. The kinetic and the potential energy per unit volume are given by

$$E_K = \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2, \qquad E_P = \rho g z. \tag{76}$$

Since it integrates to a constant, we might as well remove $\rho_0 g z$ from the potential energy. This leaves

$$E_P = (\rho - \rho_0)gz. \tag{77}$$

Now, for small-amplitude waves, $E_K = \mathcal{O}(a^2)$ but $E_P = \mathcal{O}(a)$, where $a \ll 1$ is the wave amplitude. This is odd, for a number of reasons. First, $E_K \ll E_P$, which is counter-intuitive (one expects the two forms of energy to be of the same order); second, E_P is not sign-definite; and third, the disturbance energy cannot be calculated to leading order from linear theory. To see this, consider a solution involving a perturbation expansion in some small parameter ε:

$$\rho - \rho_0 = \varepsilon\rho_1 + \varepsilon^2\rho_2 + \ldots, \qquad \boldsymbol{v} = \varepsilon\boldsymbol{v}_1 + \varepsilon^2\boldsymbol{v}_2 + \ldots. \tag{78}$$

The subscript 1 variables would be determined from linear theory, the subscript 2 variables from second-order nonlinear theory, and so on. Expanding the energies in terms of ε yields

$$E_K = \frac{1}{2}\rho_{00}|\boldsymbol{v}_1|^2\varepsilon^2 + \mathcal{O}(\varepsilon^3), \qquad E_P = \rho_1 g z\varepsilon + \rho_2 g z\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{79}$$

If we are considering sinusoidal waves then $\overline{\rho_1} = 0$ but $\overline{\rho_2} \neq 0$ in general, where the overbar denotes an average over phase. Therefore to determine E_P at leading order, $\overline{\rho_2}$ must be determined; but this requires a solution of the nonlinear problem.

All these difficulties arise from the fact that the expression for E_P is formally $\mathcal{O}(a)$. Fortunately, however, there is a remedy. Traditionally (e.g. Holliday & McIntyre 1981) it is presented as a trick. For incompressible fluids, (75) implies that $\iiint_D F(\rho)\,dxdydz$ is conserved for any function $F(\cdot)$. For a statically stable basic state $\rho_0(z)$, the inverse function $z = Z(\rho_0(z))$ is well defined. We may then take

$$F(\rho) = -\int^{\rho} gZ(\tilde{\rho})\,d\tilde{\rho}, \tag{80}$$

and note that

$$\iiint_D \Big\{E_K + E_P + F(\rho) - F(\rho_0)\Big\}\,dxdydz \tag{81}$$

is conserved. That is, we combine energy conservation with mass conservation, to obtain a new conserved quantity with density per unit volume given by

$$\begin{aligned}
A &= E_K + E_P + F(\rho) - F(\rho_0) \\
&= \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 + (\rho-\rho_0)gz - \int_{\rho_0}^{\rho} gZ(\tilde\rho)\,d\tilde\rho \\
&= \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 + (\rho-\rho_0)gz - \int_{0}^{\rho-\rho_0} gZ(\rho_0+\tilde\rho)\,d\tilde\rho \\
&= \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 - \int_{0}^{\rho-\rho_0} g\,[Z(\rho_0+\tilde\rho) - Z(\rho_0)]\,d\tilde\rho. \qquad (82)
\end{aligned}$$

The small-amplitude approximation to A (appropriate for waves, say) is

$$A \approx \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 - \frac{1}{2}gZ'(\rho_0)(\rho-\rho_0)^2 = \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 - \frac{1}{2}\frac{g}{\rho_0'(z)}(\rho-\rho_0)^2. \qquad (83)$$

The second term in (83) is the familiar expression for the APE of internal gravity waves (see e.g. Gill 1982, §6.7 or Lighthill 1978, §4.1). The conserved quantity A has the properties we would expect from a disturbance energy: $A = \mathcal{O}(a^2)$; $A > 0$ if the background is stably stratified (this is also true at finite amplitude); and A is calculable to leading order from the linearized solution. In textbooks, the small-amplitude form is derived by direct manipulation of the linearized equations — thereby obscuring the fact that mass conservation has been used.

Other cases where a similar situation arises include the energy of acoustic waves (Lighthill 1978, §1.3) and the APE of a hydrostatic compressible ideal gas (Lorenz 1955).

3.3 Pseudoenergy

When one considers the wide variety of situations in which the concept of APE arises, certain questions naturally arise. In particular: Why do other conservation laws (like mass conservation) need to be brought in? Which conservation laws are needed? Is there a systematic way to construct the APE? Does the concept extend to arbitrary fluid systems? And does it extend to non-resting basic states?

It turns out that these questions can all be answered by considering things within the Hamiltonian framework. Since fluid systems are generally non-canonical, perturbing a steady state $\boldsymbol{U}$ with a variation $\delta\boldsymbol{u}$ will give rise to a change in the Hamiltonian

$$\Delta\mathcal{H}[\boldsymbol{U};\delta\boldsymbol{u}] = \mathcal{H}[\boldsymbol{U}+\delta\boldsymbol{u}] - \mathcal{H}[\boldsymbol{U}] = \Big\langle \underbrace{\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\Big|_{\boldsymbol{u}=\boldsymbol{U}}}_{\neq 0}, \delta\boldsymbol{u}\Big\rangle + \mathcal{O}((\delta\boldsymbol{u})^2). \qquad (84)$$

This is the reason why there is an $\mathcal{O}(\delta\boldsymbol{u}) = \mathcal{O}(a)$ term in the expression for potential energy. For canonical systems, the underbraced term would vanish and the change in the Hamiltonian would automatically be quadratic in the disturbance amplitude. This is not the case here, but we know that generically there exists some Casimir $\mathcal{C}$ such that

$$\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\Big|_{\boldsymbol{u}=\boldsymbol{U}} = -\frac{\delta\mathcal{C}}{\delta\boldsymbol{u}}\Big|_{\boldsymbol{u}=\boldsymbol{U}}. \qquad (85)$$

So if we choose

$$\mathcal{A}[\boldsymbol{U};\delta\boldsymbol{u}] = \mathcal{H}[\boldsymbol{U}+\delta\boldsymbol{u}] - \mathcal{H}[\boldsymbol{U}] + \mathcal{C}[\boldsymbol{U}+\delta\boldsymbol{u}] - \mathcal{C}[\boldsymbol{U}], \tag{86}$$

with $\mathcal{C}$ determined by (85), then we will have a quantity which by construction satisfies

$$\mathcal{A}[\boldsymbol{U};0] = 0 \qquad \text{and} \qquad \left.\frac{\delta\mathcal{A}}{\delta\boldsymbol{u}}\right|_{\boldsymbol{u}=\boldsymbol{U}} = \left.\left(\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} + \frac{\delta\mathcal{C}}{\delta\boldsymbol{u}}\right)\right|_{\boldsymbol{u}=\boldsymbol{U}} = 0. \tag{87}$$

Hence $\mathcal{A}[\boldsymbol{U};\delta\boldsymbol{u}] = \mathcal{O}((\delta\boldsymbol{u})^2)$, and we have what we want.

This quantity $\mathcal{A}$ is the pseudoenergy (e.g. McIntyre & Shepherd 1987). It is an exact nonlinear invariant of the equations of motion. Its construction involves a combination of energy *and* a suitable Casimir. For disturbances to resting basic states, these Casimirs invariably involve mass conservation. The available potential energy is evidently the non-kinetic part of the pseudoenergy. To construct the available potential energy, therefore, we need only know the Hamiltonian $\mathcal{H}$; the dynamic variables, i.e. the fields $\boldsymbol{u}$; and suitable Casimirs $\mathcal{C}$ such that (85) is satisfied. One may well know these things without knowing J, in which case the Hamiltonian structure underlies the method without appearing explicitly.

Prescient adumbrations of the above realization can be found in the classical GFD literature. In a brilliant and now largely forgotten paper, Fjørtoft (1950) noted that (stably) stratified, resting basic states were energy extrema for adiabatic disturbances; this variational principle corresponds to the Hamiltonian statement that resting steady states are conditional extrema of the Hamiltonian, with the relevant Casimirs being those arising from the material conservation of entropy. Building on Fjørtoft's work, Van Mieghem (1956) used this variational principle to construct a small-amplitude expression for APE, thereby recovering the formula of Lorenz (1955). This can now be seen as the non-kinetic part of the small-amplitude (or quadratic) pseudoenergy.

Having examined this problem from the Hamiltonian standpoint, the questions raised at the beginning of this sub-section may be answered immediately.

Question: Why is energy not good enough? Why do other conservation laws (like mass conservation) need to be brought in?

Answer: Because the Eulerian descriptions of fluid motion are generally non-canonical, which means that steady states are not necessarily energy extrema.

Question: Which conservation laws are needed?

Answer: Those associated with the non-canonical nature of the dynamics: the Casimir invariants.

Question: Is there a systematic way to construct the APE?

Answer: The APE is the non-kinetic part of the pseudoenergy relative to a resting basic state.

Question: Does the concept extend to arbitrary fluid systems?

Answer: Yes, provided the system is Hamiltonian.

Question: And does it extend to non-resting basic states?

Answer: In principle, yes — provided the pseudoenergy is sign-definite. See Section 4.3 for further discussion.

3.4 Example: stratified Boussinesq flow

The algorithm described above for constructing the APE will now be demonstrated in the context of 3-D, incompressible, stratified, Boussinesq flow, governed by (74,75). The dynamical variables are evidently ρ and $\boldsymbol{v}$. The Hamiltonian is given by

$$\mathcal{H} = \iiint_D \left\{ \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 + \rho g z \right\} dxdydz, \tag{88}$$

with functional derivatives

$$\frac{\delta \mathcal{H}}{\delta \boldsymbol{v}} = \rho_{00}\boldsymbol{v}, \qquad \frac{\delta \mathcal{H}}{\delta \rho} = gz. \tag{89}$$

The determination of an appropriate J gets us into the issue of constrained dynamics (see Abarbanel *et al.* 1986; also Salmon 1988*a*), but for our purposes only the invariants $\mathcal{H}$ and $\mathcal{C}$ are required. An unconstrained Hamiltonian representation of this system, in the form (1), can be obtained by working in isentropic — or, in this case, isopycnal — coordinates (Holm & Long 1989), but for applications it is desirable to have an expression for APE in physical coordinates.

What are the Casimirs for this system? It can be verified that the potential vorticity $q = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\rho$, where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$, satisfies

$$q_t + \boldsymbol{v} \cdot \boldsymbol{\nabla} q = 0. \tag{90}$$

Putting (90) together with (75) implies that

$$\mathcal{C}[\boldsymbol{v}, \rho] = \iiint_D C(\rho, q)\, dxdydz \tag{91}$$

is a class of conserved quantities for arbitrary functions C. These are in fact the Casimirs, as can be verified by examining the system in isopycnal coordinates. However, such verification is not actually necessary: usually one can guess the Casimirs based on knowledge of the materially conserved quantities; if one cannot satisfy the condition (85), then one must consider a broader class of conserved functionals.

Taking the first variation of $\mathcal{C}$ gives

$$\begin{aligned} \delta\mathcal{C} &= \iiint_D \left\{ C_\rho \delta\rho + C_q \delta q \right\} dxdydz \\ &= \iiint_D \left\{ C_\rho \delta\rho + C_q [(\boldsymbol{\nabla} \times \delta\boldsymbol{v}) \cdot \nabla\rho + \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\delta\rho] \right\} dxdydz. \end{aligned} \tag{92}$$

After integration by parts, one obtains

$$\frac{\delta\mathcal{C}}{\delta\rho} = C_\rho - \boldsymbol{\nabla} \cdot (C_q \boldsymbol{\omega}), \qquad \frac{\delta\mathcal{C}}{\delta\boldsymbol{v}} = \boldsymbol{\nabla} \times (C_q \boldsymbol{\nabla}\rho). \tag{93}$$

We may now follow the recipe set forth earlier. Given a steady state $\boldsymbol{U} = (\boldsymbol{0}, \rho_0(z))$, $\mathcal{C}$ is determined from the condition

$$\begin{aligned} \left.\frac{\delta\mathcal{H}}{\delta\boldsymbol{v}}\right|_{\boldsymbol{u}=\boldsymbol{U}} = -\left.\frac{\delta\mathcal{C}}{\delta\boldsymbol{v}}\right|_{\boldsymbol{u}=\boldsymbol{U}} &\iff \rho_{00}\boldsymbol{v} = -\boldsymbol{\nabla} \times (C_q \boldsymbol{\nabla}\rho) \quad \text{at} \quad \rho = \rho_0,\ \boldsymbol{v} = \boldsymbol{0} \\ &\iff \boldsymbol{0} = \boldsymbol{\nabla} \times (C_q \boldsymbol{\nabla}\rho_0), \end{aligned} \tag{94}$$

which will be satisfied if we take $C = C(\rho)$, in conjunction with the condition

$$\left.\frac{\delta\mathcal{H}}{\delta\rho}\right|_{\boldsymbol{u}=\boldsymbol{U}} = -\left.\frac{\delta\mathcal{C}}{\delta\rho}\right|_{\boldsymbol{u}=\boldsymbol{U}} \iff gz = -C_\rho + \boldsymbol{\nabla}\cdot(C_q\boldsymbol{\omega}) \quad \text{at} \quad \rho=\rho_0,\, \boldsymbol{v}=\boldsymbol{0}$$
$$\iff gz = -C_\rho \quad \text{at} \quad \rho = \rho_0. \tag{95}$$

Now $\rho_0(z)$, being monotonic by hypothesis, has a well-defined inverse, Z: i.e. $z = Z(\rho_0(z))$. It can be easily seen that

$$C(\rho) = -\int^{\rho} gZ(\tilde{\rho})\, d\tilde{\rho} \tag{96}$$

satisfies the required condition (95):

$$\left. C_\rho \right|_{\rho=\rho_0} = -gZ(\rho_0) = -gz. \tag{97}$$

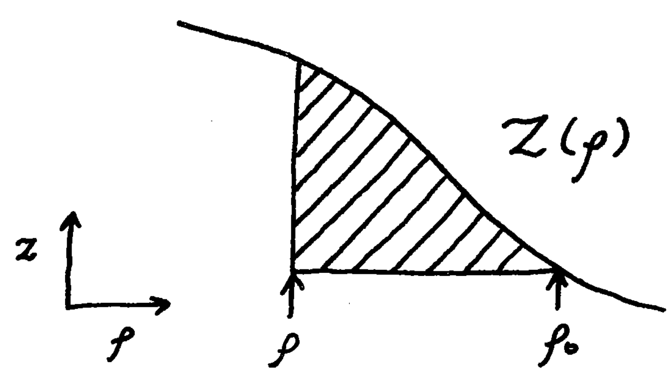
(It is in fact a general result that when the basic state is at rest, only that part of the Casimir depending on the density (or more generally, the entropy) needs to be considered; possible dependence on the potential vorticity is not required (see Shepherd 1993*a*). This is why Fjørtoft's (1950) variational principle could describe all resting steady states.)

With this choice of C, the pseudoenergy takes the form

$$\begin{aligned}
\mathcal{A} &= \mathcal{H}[\boldsymbol{v},\rho] - \mathcal{H}[\boldsymbol{0},\rho_0] + \mathcal{C}[\boldsymbol{v},\rho] - \mathcal{C}[\boldsymbol{0},\rho_0] \\
&= \iiint_D \left\{\frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 + (\rho-\rho_0)gz - \int_{\rho_0}^{\rho} gZ(\tilde{\rho})\, d\tilde{\rho}\right\} dxdydz \\
&= \iiint_D \left\{\frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 + (\rho-\rho_0)gZ(\rho_0) - \int_0^{\rho-\rho_0} gZ(\rho_0+\tilde{\rho})\, d\tilde{\rho}\right\} dxdydz \\
&= \iiint_D \left\{\frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 \underbrace{- \int_0^{\rho-\rho_0} g[Z(\rho_0+\tilde{\rho}) - Z(\rho_0)]\, d\tilde{\rho}}_{\text{APE}}\right\} dxdydz.
\end{aligned} \tag{98}$$

This recovers the expression (82) obtained earlier by direct methods. Note that provided $g(d\rho_0/dz) < 0$, then $g(dZ/d\rho_0) < 0$; thus the APE, and in consequence $\mathcal{A}$ itself, will necessarily be positive definite for $\rho - \rho_0 \neq 0$.

The finite-amplitude expression for APE provided above has a simple geometrical interpretation: the APE is g times the area under the curve $Z(\rho)$ (see below).



3.5 Nonlinear static stability

The existence of a positive definite conserved quantity suggests stability. Of course, the condition $g(d\rho_0/dz) < 0$ is precisely the condition for static stability. Note that $\delta\mathcal{A} = 0$ by construction while the second variation of $\mathcal{A}$ is

$$\delta^2\mathcal{A} = \iiint_D \frac{1}{2}\left\{\rho_{00}|\delta\boldsymbol{v}|^2 - \frac{g}{\rho_0'(z)}(\delta\rho)^2\right\} dxdydz \; > \; 0. \tag{99}$$

Thus by analogy with finite-dimensional systems, statically stable equilibria are elliptic fixed points. This is what is usually referred to as *formal stability* (e.g. Holm *et al.* 1985; see also Morrison's lectures). However, for infinite-dimensional (or continuous) dynamical systems, such as fluids, mere positivity of the second variation does not, in itself, establish anything about stability. Instead, one must attempt to obtain explicit bounds on the growth of disturbance norms. One might think it wise to begin with the linearized equations; however, if stability is established in the linear dynamics this proves nothing for the actual dynamics, since the system is Hamiltonian. (Stability can never be asymptotic for Hamiltonian systems.) Thus one is forced to consider the full nonlinear dynamics right from the start.

Definition: (Normed Stability) If we measure the deviation from a particular steady field $\boldsymbol{U}$ by the norm $||\boldsymbol{u}'||$, where $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$, then $\boldsymbol{U}$ is *stable* in that norm if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$||\boldsymbol{u}'(0)|| < \delta \quad \Longrightarrow \quad ||\boldsymbol{u}'(t)|| < \epsilon\,, \qquad \forall t \geq 0. \tag{100}$$

This is also called *Liapunov stability.*

In the present context we may define our norm by

$$||(\boldsymbol{v}, \rho - \rho_0)||^2 = \iiint_D \frac{1}{2}\left\{\rho_{00}|\boldsymbol{v}|^2 + \lambda(\rho - \rho_0)^2\right\} dxdydz, \tag{101}$$

$$\text{with} \quad c_1 \leq \lambda \leq c_2 \quad \text{where} \quad \left\{ \begin{array}{lcl} c_1 & = & \min\{-gZ'(\rho_0)\} > 0 \\ c_2 & = & \max\{-gZ'(\rho_0)\} < \infty. \end{array} \right. \tag{102}$$

The existence of such constants c_1, c_2 will be referred to as the *convexity condition.* Under these circumstances it can be shown that the available potential energy,

$$\text{APE} = -\int_0^{\rho-\rho_0} g[Z(\rho_0 + \tilde{\rho}) - Z(\rho_0)]\, d\tilde{\rho}, \tag{103}$$

is bounded from both above and below:

$$\frac{1}{2}c_1(\rho - \rho_0)^2 \leq \; \text{APE} \; \leq \frac{1}{2}c_2(\rho - \rho_0)^2. \tag{104}$$

If $Z(\rho_0)$ is smooth then this result follows immediately from Taylor's remainder theorem. However, it is true for $Z(\rho_0)$ that are only piecewise differentiable. This bound on the APE, coupled with the fact that the pseudoenergy $\mathcal{A}$ is conserved, leads to the following chain of inequalities for basic flows satisfying the convexity condition (102):

$$\begin{aligned} \|(\boldsymbol{v}, \rho - \rho_0)(t)\|^2 &= \iiint_D \frac{1}{2}\left\{\rho_{00}|\boldsymbol{v}|^2 + \lambda(\rho-\rho_0)^2\right\}(t)\, dxdydz \\ &\le \frac{\lambda}{c_1}\mathcal{A}(t) = \frac{\lambda}{c_1}\mathcal{A}(0) \le \frac{c_2}{c_1}\|(\boldsymbol{v}, \rho-\rho_0)(0)\|^2. \end{aligned} \tag{105}$$

By taking $\delta = \sqrt{c_1/c_2}\epsilon$, (105) proves nonlinear normed stability in the norm defined by (101).

It is well known that the small-amplitude definition of APE is closely connected to linearized static stability. The above results show that the definition of *finite-amplitude* APE is closely connected to *finite-amplitude* static stability.

A comment should be made concerning the definition of the function $Z(\cdot)$ used to calculate the pseudoenergy, which appears in the integrand of expression (103) for the APE. What if ρ lies outside the range of ρ_0? How do we evaluate $Z(\rho)$ in that case? First note that if a disturbance is "dynamically accessible" (see Morrison's lectures) then ρ always lies within the range of ρ_0. However if one is interested in a larger class of disturbances, then the definition of $Z(\rho)$ can be extended outside the range of ρ_0 while still keeping $\mathcal{A}$ as a conserved quantity. This is because *any* function $C(\rho)$ can be used to obtain a Casimir. In fact, it is only the condition that $\mathcal{A} = \mathcal{O}(a^2)$ in the small-amplitude limit that determined the particular choice of C involving Z, and this constraint only determines C for values of its argument lying within the range of ρ_0. So to allow the possibility of arbitrary disturbances, the expression (103) can still be used provided we extend the function $Z(\rho)$ outside the range of ρ_0 in some arbitrary way, subject only to

$$c_1 \le -gZ'(\rho) \le c_2 \tag{106}$$

in order not to weaken our bounds. Clearly, this extension can always be made.

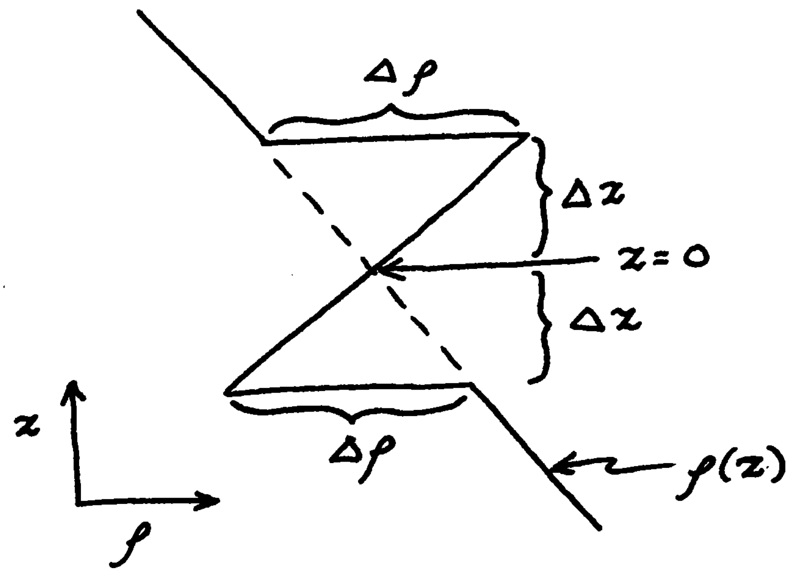
3.6 Nonlinear saturation of instabilities

The APE provides a rigorous upper bound on the saturation of static instabilities. In a way, this is a more robust definition of static stability than the concept of normed stability presented in the previous sub-section.

To see this, consider the case of a fluid that is initially at rest but statically unstable. We may consider this initial state to be a (finite-amplitude) disturbance to some statically stable, resting basic state. Using conservation of the pseudoenergy relative to this basic state, and noting that $\boldsymbol{v}(t=0) = \mathbf{0}$ by hypothesis, yields

$$\iiint_D \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2(t)\, dxdydz \le \mathcal{A}(t) = \mathcal{A}(0) = \iiint_D APE(0)\, dxdydz. \tag{107}$$

Thus the kinetic energy at any time t is bounded from above by the initial APE: this is, after all, why it was called "available" by Lorenz. For example, consider the situation sketched below.



The initial density profile $\rho(z)$ is shown by the solid line, and consists of a statically unstable inversion layer $-\Delta z \leq z \leq \Delta z$ located within a stably stratified region with $d\rho/dz = -r$. Suppose for definiteness that $\rho(z=0) = 0$. (Recall that ρ is the departure from the reference state ρ_{00}, and therefore may be negative.) One may choose as the basic state $\rho_0(z) = -rz$, which is stable and which satisfies (102) with $c_1 = c_2 = g/r$. The initial disturbance is then

$$\rho' = \rho - \rho_0 = \begin{cases} \dfrac{\Delta\rho}{\Delta z}\, z, & \text{if } -\Delta z \leq z \leq \Delta z \\ 0, & \text{otherwise .} \end{cases} \tag{108}$$

The integrated APE (averaged in x and y) is then easily computed, yielding the saturation bound

$$\iiint_D \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2(t)\, dxdydz \leq \frac{g}{3r}\,(\Delta\rho)^2\Delta z. \tag{109}$$

It is interesting to note here that the disturbance ρ' is not, in general, dynamically accessible; or, rather, the initial condition $\rho(t=0)$ is not dynamically accessible from the basic state $\rho_0(z)$. It would only be so in the special case $\Delta\rho/\Delta z = r$. The bound represented by (109) therefore highlights the fact that pseudoenergy conservation holds for *arbitrary* disturbances, not just dynamically accessible ones. It also demonstrates that, in many practical cases of interest, the freedom to consider disturbances that are not dynamically accessible is quite useful. The original, physical definition of APE proposed by Margules (1903) and formalized by Lorenz (1955) was keyed around the idea of dynamically accessible perturbations: the APE was defined to be the amount of energy released in an adiabatic rearrangement of the mass into a statically stable state. The variational approach of Fjørtoft (1950) and Van Mieghem (1956) likewise builds dynamical accessibility directly into the theory. In contrast, the use of integral invariants, in particular the pseudoenergy, goes beyond this in a powerful way: the basic state may be *any* statically stable state, not just the dynamically accessible one. This insight was first noted by Holliday & McIntyre (1981) and Andrews (1981), and is vividly demonstrated by the above example.

One may logically define the APE in (107) to be the minimum APE over all possible choices of the stable basic state. This extremization problem is highly non-trivial, and would make a good topic for further study from a mathematical perspective. In the above example, for instance, the expression (109) is merely one bound, not necessarily the minimum one.

Exercise: Calculate the amount of APE in the initial condition of the above example, taking the basic state $\rho_0(z)$ to be the unique, dynamically accessible state obtained through an adiabatic rearrangement of the mass.

There is a well-known analogy between static stability and so-called *symmetric stability*: namely the stability of a baroclinic flow to disturbances that do not vary in the downstream direction (also known as "slantwise convection"). This analogy has recently been exploited by Cho, Shepherd & Vladimirov (1993), who prove a nonlinear stability theorem and use it to determine a finite-amplitude APE for such motion.

4 Pseudoenergy and Arnol'd's stability theorems

4.1 Arnol'd's stability theorems

In the previous section the steady basic state was at rest, so the kinetic energy contribution to the pseudoenergy was solely the disturbance kinetic energy. But what happens when the kinetic energy of the steady state is nonzero? To explore this question, we study the barotropic vorticity equation on the β-plane (cf. §1.4)

$$P_t + \partial(\Phi, P) = 0, \tag{110}$$

where Φ is the stream function, P is the potential vorticity

$$P = \nabla^2\Phi + f + \beta y + h(x, y), \tag{111}$$

and h is the topographic height. Three possible geometries are considered: (i) periodic in x and y; (ii) unbounded, with decay conditions at infinity; and (iii) multiple boundaries (as with a zonal channel). The last case is the most complicated, since the boundary terms enter the equations directly, so we choose to analyze it.

Suppose there exists a steady solution, $\Phi = \Psi$, $P = Q$ with $\Psi = \Psi(Q)$ a monotonic function. We seek Casimirs such that $\delta\mathcal{A} = 0$. We have

$$\mathcal{H} = \iint_D \frac{1}{2}|\nabla\Phi|^2\, dxdy, \tag{112}$$

$$\mathcal{C} = \iint_D C(P)\, dxdy + \sum_i a_i\mu_i, \tag{113}$$

where $\mu_i \equiv \oint_{\partial D_i} \nabla\Phi \cdot \hat{\mathbf{n}}\, ds$ is the circulation on each connected piece, ∂D_i, of the boundary ∂D. To determine the pseudoenergy, we must solve the equations

$$\left.\frac{\delta\mathcal{H}}{\delta P}\right|_{P=Q} = -\left.\frac{\delta\mathcal{C}}{\delta P}\right|_{P=Q}, \qquad \left.\frac{\delta\mathcal{H}}{\delta\mu_i}\right|_{P=Q} = -\left.\frac{\delta\mathcal{C}}{\delta\mu_i}\right|_{P=Q}. \tag{114}$$

The left half of (114) gives (see §1.4)

$$\Psi = C'(Q), \tag{115}$$

which integrates to

$$C(Q) = \int^{Q} \Psi(\eta)d\eta, \tag{116}$$

while the right half of (114) gives

$$\left.\Psi\right|_{\partial D_i} = -a_i. \tag{117}$$

If we consider the disturbance problem

$$P = Q + q, \quad \Phi = \Psi + \psi, \quad \mu_i = \Gamma_i + \gamma_i, \tag{118}$$

then noting that $q = \nabla^2\psi$ we construct the finite-amplitude pseudoenergy as follows:

$$\begin{aligned}
\mathcal{A} &= \mathcal{H}[Q+q,\Gamma_i+\gamma_i] - \mathcal{H}[Q,\Gamma_i] + \mathcal{C}[Q+q,\Gamma_i+\gamma_i] - \mathcal{C}[Q,\Gamma_i] \\
&= \iint_D \Big\{\nabla\Psi\cdot\nabla\psi + \frac{1}{2}|\nabla\psi|^2 + \int_Q^{Q+q} \Psi(\tilde{q})\, d\tilde{q}\Big\}\, dxdy + \sum_i a_i\gamma_i \\
&= \iint_D \Big\{\nabla\cdot(\Psi\nabla\psi) - \Psi\nabla^2\psi + \frac{1}{2}|\nabla\psi|^2 + \int_0^{q} \Psi(Q+\tilde{q})\, d\tilde{q}\Big\}\, dxdy - \sum_i \left.\Psi\right|_{\partial D_i}\gamma_i
\end{aligned} \tag{119}$$

In the last line, the first term can be directly integrated and found to cancel the boundary circulation terms. Furthermore, noting that

$$-\Psi\nabla^2\psi = -\Psi q = -\int_0^{q} \Psi(Q), \tag{120}$$

the pseudoenergy can be written simply as

$$\mathcal{A} = \iint_D \Big\{\frac{1}{2}|\nabla\psi|^2 + \int_0^{q} [\Psi(Q+\tilde{q}) - \Psi(Q)]\, d\tilde{q}\Big\}\, dxdy \tag{121}$$

(McIntyre & Shepherd 1987). The pseudoenergy is an exact, nonlinear invariant, as may be checked by direct substitution into the equations of motion. It is valid for *arbitrary* disturbances (not necessarily dynamically accessible ones). If there exist values of $Q+q$ outside the range of Q, one can extend the definition of $\Psi(Q)$ arbitrarily to cover those values, as discussed in §3.5. $\mathcal{A}$ is evidently sign-definite when

$$\frac{d\Psi}{dQ} > 0. \tag{122}$$

Essentially, this is Arnol'd's (1966) first stability theorem.

Suppose that

$$c_1 = \min_Q \left\{ \frac{d\Psi}{dQ} \right\} > 0, \qquad c_2 = \max_Q \left\{ \frac{d\Psi}{dQ} \right\} < \infty. \tag{123}$$

In this case we can establish normed stability of the basic state. The "convexity" condition provides

$$\frac{1}{2} c_1 q^2 \le \int_0^q \left[\Psi(Q + \tilde{q}) - \Psi(Q) \right] d\tilde{q} \le \frac{1}{2} c_2 q^2, \tag{124}$$

which is valid for continuous (possibly non-smooth) $\Psi(Q)$. In particular, let us choose the norm defined by

$$||q||^2 = \iint_D \frac{1}{2} \left\{ |\nabla\psi|^2 + \lambda q^2 \right\} dxdy, \tag{125}$$

with $c_1 \le \lambda \le c_2$. Then

$$||q(t)||^2 \le \frac{\lambda}{c_1} \mathcal{A}(t) = \frac{\lambda}{c_1} \mathcal{A}(0) \le \frac{c_2}{c_1} ||q(0)||^2. \tag{126}$$

So given $\epsilon > 0$, choose $\delta = \sqrt{c_1/c_2}\epsilon$ to prove nonlinear normed stability. As with static stability, this holds for *arbitrarily* large disturbances.

It is important to emphasize that the demonstration of normed stability provided above depends on the choice of norm. For normed stability, it is *always* essential to specify the norm; this is because in infinite-dimensional spaces, all norms are not equivalent. This point is highlighted by the following example.

Consider (110) in the special case $P = \nabla^2\Phi$, and introduce a basic state $U(y) = \lambda y$. The disturbance (q, ψ) is given by

$$P = Q + q = -\lambda + q, \qquad \Phi = \Psi + \psi = -\frac{1}{2}\lambda y^2 + \psi, \tag{127}$$

and the exact equation for the disturbance vorticity $q = \nabla^2\psi$ is

$$q_t = -\partial(\Psi, q) - \partial(\psi, Q) - \partial(\psi, q) = -\lambda y q_x - \partial(\psi, q). \tag{128}$$

Multiplying (128) by q and integrating over the domain yields

$$\begin{aligned} \frac{d}{dt} \iint_D \frac{1}{2} q^2 \, dxdy &= -\iint_D \lambda y q q_x \, dxdy - \iint_D q \partial(\psi, q) \, dxdy \\ &= -\iint_D \frac{\partial}{\partial x}\left(\frac{1}{2}\lambda y q^2\right) dxdy - \iint_D \partial(\psi, \frac{1}{2} q^2) \, dxdy \\ &= 0. \end{aligned} \tag{129}$$

This proves that 2-D linear shear flow is stable in the *enstrophy* norm. However, consider an initial condition consisting of a plane wave $q(t = 0) = \Re\{e^{i(kx + l_0 y)}\}$. Then the disturbance energy is given by (e.g. Shepherd 1985)

$$\mathcal{E}(t) = \iint_D \frac{1}{2} |\nabla\psi|^2(t) \, dxdy = \frac{\iint_D \frac{1}{2} q^2(t) \, dxdy}{k^2 + l^2} = \frac{\iint_D \frac{1}{2} q^2(0) \, dxdy}{k^2 + l^2}. \tag{130}$$

It can be shown that l evolves with time according to $l = l_0 + \lambda k t$, while k is constant. So for $l_0 < 0$ and $\lambda k > 0$, the energy will attain its maximum amplification

$$\frac{\mathcal{E}(t)}{\mathcal{E}(0)} = \frac{k^2 + l_0^2}{k^2} \qquad \text{at} \qquad t = \frac{-l_0}{k\lambda}. \tag{131}$$

Clearly, the amplification described by (131) becomes arbitrarily large in the limit $|l_0| \to \infty$. This example demonstrates the point that stability in one norm (here the enstrophy) does not imply stability in another (here the energy).

Returning to the general form of (110), consider the special case of zonal (x-invariant) flow, with $h = 0$. Then the condition

$$\frac{d\Psi}{dQ} = \frac{\nabla\Psi}{\nabla Q} = \frac{\Psi_y}{Q_y} = \frac{-U}{Q_y} > 0 \tag{132}$$

is sufficient for stability of the flow. This is the nonlinear generalization of the result of Fjørtoft (1950).

There is an interesting possibility in this barotropic case which did not arise in the previously discussed case of static stability. Recall from §3.5 that in that case the pseudoenergy was given by

$$\mathcal{A} = \iiint_D \left\{ \frac{1}{2}\rho_{00}|\boldsymbol{v}|^2 + \mathrm{APE}(\rho - \rho_0) \right\} dx\,dy\,dz. \tag{133}$$

Since ρ and $\boldsymbol{v}$ are independent variables, $\mathcal{A}$ can never be negative definite. This is like the case of "natural systems" discussed in traditional classical mechanics. In the present case, however, there is only one dynamical variable, and in principle $\mathcal{A}$ *could* be negative definite. This gives what is called Arnol'd's (1966) second stability theorem.

How does this happen? Suppose

$$\frac{d\Psi}{dQ} < 0 \quad \text{and} \quad c_1 = \min\left\{-\frac{d\Psi}{dQ}\right\} > 0, \quad c_2 = \max\left\{-\frac{d\Psi}{dQ}\right\} < \infty. \tag{134}$$

Then

$$\int_0^q [\Psi(Q + \tilde{q}) - \Psi(Q)]\, d\tilde{q} \le -\frac{1}{2}c_1 q^2, \tag{135}$$

so this quantity has the potential for being more negative than $\frac{1}{2}|\nabla\psi|^2$ is positive, when integrated over the domain. In fact, for bounded domains this is possible. A detailed discussion is provided in McIntyre & Shepherd (1987, §6).

4.2 Andrews' theorem

The appearance of Arnol'd-type stability arguments created considerable interest in the meteorological community, for it appeared that they could be used to examine the stability of non-parallel flow profiles $\Psi = \Psi(Q)$. However, Arnol'd's theorems turn out to be not as powerful as they might seem in this regard. A theorem proved by Andrews (1984) shows this quite succinctly, as follows.

Suppose we are given a basic flow profile $\Psi = \Psi(Q)$, and suppose that the given problem is zonally symmetric: i.e. $h = h(y)$, and the boundaries (if any) are independent of x. A zonal channel would be the most common such geometry.

Claim: If $\dfrac{d\Psi}{dQ} > 0$, then $Q_x = 0$ and $\Psi_x = 0$.

Proof: The chain rule of differentiation implies

$$\Psi_x = \Psi'(Q) Q_x. \tag{136}$$

Multiplying this expression by Q_x and integrating over the domain yields

$$\begin{aligned} \iint_D \Psi_x Q_x \, dxdy &= \iint_D \Psi'(Q) \, (Q_x)^2 \, dxdy \\ \Longleftrightarrow \iint_D \Psi_x \nabla^2 \Psi_x \, dxdy &= \iint_D \Psi'(Q) \, (Q_x)^2 \, dxdy \\ \Longleftrightarrow \iint_D \boldsymbol{\nabla} \cdot (\Psi_x \boldsymbol{\nabla} \Psi_x) \, dxdy &= \iint_D \left\{ |\boldsymbol{\nabla} \Psi_x|^2 + \Psi'(Q) \, (Q_x)^2 \right\} dxdy. \end{aligned} \tag{137}$$

The integral on the left-hand side vanishes if the boundaries are zonally symmetric, which implies that $\Psi_x = 0 = Q_x$ everywhere.

Therefore, any flow in a zonally symmetric domain that is stable by Arnol'd's first theorem must itself be zonally symmetric! The argument can also be shown to apply to Arnol'd's second theorem (Carnevale & Shepherd 1990). These results help explain the conspicuous lack of non-zonal Arnol'd-stable flows in the literature.

There is a simple heuristic explanation of Andrews' theorem. If a problem is zonally symmetric, but the basic state is non-zonal, then a possible disturbance is the simple one generated by a zonal translation of the basic state. This zonal translation cannot change the pseudoenergy. Therefore, such basic states cannot be true extrema of the pseudoenergy — equivalently, $\mathcal{A}$ is not sign-definite — which implies that they cannot be Arnol'd stable.

However, it should be noted that Andrews' theorem may not apply to certain zonally symmetric problems in unbounded domains because of the boundary conditions at infinity (Carnevale & Shepherd 1990). Otherwise one could deduce, for example, that circular vortices were not Arnol'd stable — something which is demonstrably untrue.

4.3 Available energy

Can we regard the quantity

$$\int_0^q [\Psi(Q + \tilde{q}) - \Psi(Q)] \, d\tilde{q} \tag{138}$$

as a generalization of APE? In a sense, yes. For any stable flow (Ψ, Q) with $d\Psi/dQ > 0$, we have

$$\iint_D \frac{1}{2} |\boldsymbol{\nabla} \psi|^2(t) \, dxdy \le \mathcal{A}(t) = \mathcal{A}(0) = \mathcal{A}[Q; q(0)]. \tag{139}$$

For a given initial condition, $P(0)$, one can vary the right-hand side of (139) over various stable Q to seek the tightest possible bound on the disturbance energy.

In the case of static stability, the variations were over ρ_0 and didn't affect $\frac{1}{2}|\boldsymbol{v}|^2$. Here $\psi = \Phi - \Psi$, so when one changes the basic flow, one also changes the definition of the disturbance. This is not a satisfactory situation.

Suppose, however, that the problem is zonally symmetric and that one is interested in the eddy energy. We may write

$$\Phi = \overline{\Phi} + \Phi' \text{ where } \overline{(\ )} \equiv x\text{-average and } \overline{\Phi'} = 0. \tag{140}$$

Then, if we choose a zonally symmetric flow (as required by Andrews' theorem), i.e.

$$\Psi' = 0, \qquad Q' = 0, \tag{141}$$

this implies

$$\Phi' = \psi', \qquad P' = q'. \tag{142}$$

Hence (q', ψ') are independent of the choice of the basic state. Then, for the eddy energy we have the upper bound

$$\begin{aligned}\mathcal{E}' \equiv \iint_D \frac{1}{2}|\nabla\Phi'|^2(t)\, dxdy = \iint_D \frac{1}{2}|\nabla\psi'|^2(t)\, dxdy \;&\leq\; \iint_D \frac{1}{2}|\nabla\psi|^2(t)\, dxdy \\ &\leq\; \mathcal{A}[Q; q(0)].\end{aligned} \tag{143}$$

Now one can vary the right-hand side of (143) to seek the tightest possible bound on the eddy energy.

We now illustrate the general method by applying it to the case of baroclinic flow.

4.4 Nonlinear saturation of baroclinic instability

The two-layer model was presented in §2.1. The notation has been changed somewhat for convenience; q there corresponds to P here, while ψ there corresponds to Φ here. Further details of the following analysis may be found in Shepherd (1993*b*).

Suppose $F_1 = F_2 = F$ in the domain $0 \leq y \leq 1$, periodic in x. The potential vorticity in each layer is given by

$$P_i = \nabla^2\Phi_i + (-1)^i F(\Phi_1 - \Phi_2) + f + \beta y \quad [i = 1, 2]. \tag{144}$$

Consider the basic-state stream function $\Psi_i = \Psi_i(Q_i)$ corresponding to the purely zonal flow

$$U_i(y) = -d\Psi_i/dy \quad [i = 1, 2], \tag{145}$$

with potential vorticity

$$Q_i(y) = \nabla^2\Psi_i + (-1)^i F(\Psi_1 - \Psi_2) + f + \beta y \quad [i = 1, 2]. \tag{146}$$

Let ψ_i be the disturbance stream function, so that

$$\Phi_i = \Psi_i(y) + \psi_i(x, y, t) \quad [i = 1, 2]. \tag{147}$$

This allows the pseudoenergy to be written

$$\begin{aligned}\mathcal{A} \;=\; \iint_D \Bigg\{ &\frac{1}{2}\left[|\nabla\psi_1|^2 + |\nabla\psi_2|^2 + F(\psi_1 - \psi_2)^2\right] \\ &+ \int_0^{q_1} [\Psi_1(Q_1 + \tilde{q}) - \Psi_1(Q_1)]\, d\tilde{q} + \int_0^{q_2} [\Psi_2(Q_2 + \tilde{q}) - \Psi_2(Q_2)]\, d\tilde{q} \Bigg\}\, dxdy,\end{aligned} \tag{148}$$

where q_i is the disturbance potential vorticity

$$q_i = \nabla^2\psi_i + (-1)^i F(\psi_1 - \psi_2) \quad [i = 1, 2]. \tag{149}$$

It is clear from (148) that if

$$\frac{d\Psi_1}{dQ_1} > 0 \qquad \text{and} \qquad \frac{d\Psi_2}{dQ_2} > 0, \tag{150}$$

then $\mathcal{A} > 0$. This is Arnol'd's first stability theorem applied to quasi-geostrophic flow (Holm *et al.* 1985). Since we are considering the zonally symmetric case, these conditions are satisfied if

$$\frac{U_i}{dQ_i/dy} < 0 \quad [i = 1, 2]; \tag{151}$$

put this way, the theorem represents the nonlinear version of the Fjørtoft-Pedlosky theorem.

Suppose our initial condition consists of an infinitesimal disturbance to the Phillips zonal flow

$$-\frac{\partial\overline{\Phi}_1}{\partial y} = \hat{U}_1 = \frac{\beta}{F}(1 + \epsilon) + u_0, \qquad -\frac{\partial\overline{\Phi}_2}{\partial y} = \hat{U}_2 = u_0, \tag{152}$$

where u_0 is an arbitrary constant. The flow (152) is known to be unstable if $\epsilon > 0$, provided the domain is sufficiently wide in an appropriate sense (e.g. Pedlosky 1987, §7.11).

Now choose a one-parameter family of stable basic flows

$$-\frac{\partial\Psi_1}{\partial y} = U_1 = \frac{\beta}{F}(1 - \delta) + u_0, \qquad -\frac{\partial\Psi_2}{\partial y} = U_2 = u_0, \tag{153}$$

with associated potential vorticity

$$Q_1 = \beta(2 - \delta)(y - \lambda) + f + \beta\lambda, \qquad Q_2 = \beta\delta(y - \lambda) + f + \beta\lambda, \tag{154}$$

where λ is a constant of integration. We have three free parameters: λ, u_0, and δ. For all λ, u_0, and δ such that $d\Psi_i/dQ_i > 0$ we then have the rigorous upper bound on the eddy energy

$$\begin{aligned}
\mathcal{E}' &\equiv \iint_D \frac{1}{2}\left\{|\nabla\Phi_1'|^2 + |\nabla\Phi_2'|^2 + F(\Phi_1' - \Phi_2')^2\right\} dxdy \\
&\le \iint_D \frac{1}{2}\left\{|\nabla\psi_1|^2 + |\nabla\psi_2|^2 + F(\psi_1 - \psi_2)^2\right\} dxdy \\
&\le \mathcal{A}(t) \\
&= \mathcal{A}(0) \\
&= \frac{\beta^2(\epsilon + \delta)^2}{F^2}\left\{\frac{1}{2}\Big[1 + F\int_0^1 (y - \lambda)^2 dy\Big]\right. \\
&\quad \left. -\frac{1}{2}\Big[\frac{\beta(1 - \delta) + u_0 F}{\beta F(2 - \delta)} + \frac{u_0}{\beta\delta}\Big] F^2 \int_0^1 (y - \lambda)^2 dy\right\} \\
&\quad + \text{terms involving the initial non-zonal disturbance.}
\end{aligned} \tag{155}$$

The contribution to the initial pseudoenergy associated with the initial non-zonal disturbance can, of course, be included in the above bound. However, in the situation we are considering here of an infinitesimal initial disturbance to the Phillips initial condition, this contribution is negligible compared with the pseudoenergy arising from the zonal-mean part of the initial disturbance to the stable basic flow, which is the part written out explicitly in the last line of (155).

Now, choosing $\lambda = \frac{1}{2}$ so that

$$\int_0^1 (y-\lambda)^2 dy = \frac{1}{12}, \tag{156}$$

which is the best choice, and setting

$$\beta(1-\delta) + u_0 F = 0 \qquad \Longrightarrow \qquad u_0 = -\frac{\beta(1-\delta)}{F}, \tag{157}$$

leaves

$$\mathcal{A}(0) = \frac{\beta^2(\epsilon+\delta)^2}{2F^2}\Big[1 + \frac{F}{12} + \frac{(1-\delta)F}{12\delta}\Big] = \frac{\beta^2(\epsilon+\delta)^2}{24F^2}\Big[12 + \frac{F}{\delta}\Big]. \tag{158}$$

Setting $\partial A(0)/\partial\delta = 0$ yields a minimum at

$$\delta = \frac{F}{48}\Big[-1 + \sqrt{1 + (96\epsilon/F)}\Big] \simeq \epsilon \qquad \text{for} \qquad \epsilon \ll 1. \tag{159}$$

One could, of course, use the optimal value of δ, but the simple choice $\delta = \epsilon$ certainly gives a valid saturation bound too, which is

$$\mathcal{E}' \le \mathcal{A}(0) = \frac{\beta^2}{6F}\Big(1 + \frac{12\epsilon}{F}\Big)\epsilon. \tag{160}$$

Now the eddy energy is also bounded by the total energy of the system, namely

$$\mathcal{E}' \le \mathcal{E}_{\text{total}} = \frac{\beta^2}{24F}\Big(1 + \frac{6}{F}\Big)(1+\epsilon)^2. \tag{161}$$

However it is clear that $\mathcal{E}_{\text{total}} \gg \mathcal{A}(0)$ for $\epsilon \ll 1$, so the bound (160) is providing a non-trivial constraint on the dynamics.

This gives a bound on the scaling of the saturation amplitude of the instability. But is it any good? Weakly-nonlinear theory (Pedlosky 1970; Warn & Gauthier 1989) gives

$$\mathcal{E}'_{\text{max}} \sim \frac{\beta^2\epsilon}{F} \tag{162}$$

for $\epsilon \ll 1$, which is the *same* scaling as (160); the numerical factors in (162) are $1/\pi^2$ for the non-resonant case, and $1/8$ for the resonant case. This is to be compared with a coefficient of $1/6$ for the stability-based bound. So the bound is, in fact, not too bad as an estimate of the saturation amplitude.

An important generalization of these saturation bounds is to forced-dissipative systems. If

$$\frac{DP}{Dt} = -r(P - P_e) \tag{163}$$

(potential vorticity relaxation), where $P_e = \overline{P}(t=0)$, then these bounds on the eddy energy remain valid (see Shepherd 1988*b*).

5 Pseudomomentum

5.1 General construction

We have seen how one can construct the pseudoenergy, a second-order invariant, for disturbances to a *steady* basic state under *time-invariant* dynamics. Similarly, if the dynamics under consideration is invariant to translations in the x direction, we may construct a second-order invariant for disturbances to x*-invariant* basic states.

Recall how we constructed the pseudoenergy from the Hamiltonian, $\mathcal{H}$, and the Casimirs, $\mathcal{C}$: since the basic state $\boldsymbol{U}$ in that case is a steady solution of the dynamics,

$$J \left. \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} \right|_{\boldsymbol{u}=\boldsymbol{U}} = 0. \tag{164}$$

It follows that for some Casimir $\mathcal{C}$,

$$\left. \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} \right|_{\boldsymbol{u}=\boldsymbol{U}} = - \left. \frac{\delta \mathcal{C}}{\delta \boldsymbol{u}} \right|_{\boldsymbol{u}=\boldsymbol{U}}. \tag{165}$$

The pseudoenergy is then

$$\mathcal{A}[\boldsymbol{U}; \delta\boldsymbol{u}] = \mathcal{H}[\boldsymbol{U} + \delta\boldsymbol{u}] - \mathcal{H}[\boldsymbol{U}] + \mathcal{C}[\boldsymbol{U} + \delta\boldsymbol{u}] - \mathcal{C}[\boldsymbol{U}], \tag{166}$$

where $\mathcal{C}$ is defined by (165).

In the very same way, we may construct the *pseudomomentum* from the momentum invariant, $\mathcal{M}$. By definition of $\mathcal{M}$ (see §1.5), $J(\delta\mathcal{M}/\delta\boldsymbol{u}) = -\boldsymbol{u}_x$. Now, since the Hamiltonian is presumed to be invariant under translations in the x direction, it follows from Noether's theorem that $\mathcal{M}$ is an integral of the motion. If the basic state $\boldsymbol{U}$ is also invariant with respect to x translations, then

$$J \left. \frac{\delta \mathcal{M}}{\delta \boldsymbol{u}} \right|_{\boldsymbol{u}=\boldsymbol{U}} = -\boldsymbol{U}_x = 0. \tag{167}$$

It follows that there exists a $\mathcal{C}$ such that

$$\left. \frac{\delta \mathcal{M}}{\delta \boldsymbol{u}} \right|_{\boldsymbol{u}=\boldsymbol{U}} = - \left. \frac{\delta \mathcal{C}}{\delta \boldsymbol{u}} \right|_{\boldsymbol{u}=\boldsymbol{U}}. \tag{168}$$

Finally we define the pseudomomentum by

$$\mathcal{A}[\boldsymbol{U}; \delta\boldsymbol{u}] = \mathcal{M}[\boldsymbol{U} + \delta\boldsymbol{u}] - \mathcal{M}[\boldsymbol{U}] + \mathcal{C}[\boldsymbol{U} + \delta\boldsymbol{u}] - \mathcal{C}[\boldsymbol{U}], \tag{169}$$

where $\mathcal{C}$ is defined by (168) so that $(\delta\mathcal{A}/\delta\boldsymbol{u})|_{\boldsymbol{u}=\boldsymbol{U}} = 0$. In fact, it is clear from Noether's theorem that we may generate a similar functional for *any* continuous symmetry of the dynamics.

The pseudomomentum, like the pseudoenergy, is guaranteed to have the following nice properties: (i) it is calculable to leading order from linear theory; (ii) it may be sign-definite under certain conditions. If we find some zonal basic states for which the pseudomomentum *is* sign definite, then it is clear that we are in a position to generate more nonlinear stability theorems.

If the basic flow is both zonally symmetric and steady (as zonally symmetric flows often are), we may combine the pseudoenergy and the pseudomomentum to generate still more quadratic invariants, according to

$$\mathcal{A} = (\mathcal{H} + \alpha\mathcal{M} + \mathcal{C})[\boldsymbol{u}] - (\mathcal{H} + \alpha\mathcal{M} + \mathcal{C})[\boldsymbol{U}]. \tag{170}$$

Here we may choose α arbitrarily and, again, $\mathcal{C}$ is chosen so that $(\delta\mathcal{A}/\delta\boldsymbol{u})|_{\boldsymbol{u}=\boldsymbol{U}} = 0$.

5.2 Example: Barotropic vorticity equation

In this section we will develop an expression for the pseudomomentum of the barotropic model in a β-plane zonal channel. The flows we will consider are governed by the vorticity equation

$$P_t + \partial(\Phi, P) = 0 \tag{171}$$

where Φ is the stream function, and P is the absolute vorticity

$$P = \nabla^2\Phi + f + \beta y. \tag{172}$$

For definiteness, we consider flows that are periodic in the x (zonal) direction, and bounded by rigid walls in the y direction.

We take as the state variable the absolute vorticity, P, and the boundary circulations, $\mu_i \equiv \oint_{\partial D_i} \nabla\Phi \cdot \hat{\mathbf{n}}\, ds$. Recall that in this formulation, the Hamiltonian is given by

$$\mathcal{H} = \iint_D \frac{1}{2}|\nabla\Phi|^2\, dxdy. \tag{173}$$

The cosymplectic operator, J, acting on the basis $(P,\, \mu_1,\, \mu_2)$, is given by

$$J = \begin{pmatrix} -\partial(P,\cdot) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{174}$$

The Casimirs associated with J are functionals of the form

$$\mathcal{C} = \iint_D C(P)\, dxdy + \sum_{i=1}^{2} a_i\mu_i \tag{175}$$

for arbitrary functions $C(\cdot)$ and scalars a_i. The momentum invariant $\mathcal{M}$ is found by solving

$$-\Big(\frac{\partial P}{\partial x}, 0, 0\Big)^T = J\Big(\frac{\delta\mathcal{M}}{\delta P}, \frac{\delta\mathcal{M}}{\delta\mu_1}, \frac{\delta\mathcal{M}}{\delta\mu_2}\Big)^T \qquad \Longrightarrow \qquad \frac{\partial P}{\partial x} = \partial\Big(P, \frac{\delta\mathcal{M}}{\delta P}\Big). \tag{176}$$

The solution (to within a Casimir) of (176) is given by

$$\frac{\delta\mathcal{M}}{\delta P} = y \qquad \Longrightarrow \qquad \mathcal{M} = \iint_D yP\, dxdy. \tag{177}$$

This expression differs from the usual definition of momentum, which is $\iint_D u\,dxdy$. The reader may verify (cf. §1.5) that the difference between these two expressions can be written solely as a function of the boundary circulations μ_1 and μ_2 — in other words, the difference is a Casimir.

If the basic state is given by $\Phi = \Psi$, $P = Q$, then to find the pseudomomentum we must solve

$$\left.\frac{\delta\mathcal{M}}{\delta P}\right|_{P=Q} = -\left.\frac{\delta\mathcal{C}}{\delta P}\right|_{P=Q} \qquad \Longrightarrow \qquad y = -C'(Q). \tag{178}$$

Note that this requires Q to be independent of x, i.e. $Q = Q(y)$. Thus zonally symmetric states are seen to be constrained extrema of $\mathcal{M}$, just as steady states are constrained extrema of $\mathcal{H}$. Solving (178) for C yields

$$C(Q) = -\int^{Q} Y(\tilde{q})\,d\tilde{q}, \tag{179}$$

where $Y(\cdot)$ is the inverse of $Q(y)$: that is, $y = Y(Q(y))$.

Note that since the disturbance need not be dynamically accessible, we may (as before) extend the definition of $Y(\cdot)$, if required, to cover values of its argument outside the range of the basic state Q.

The pseudomomentum, $\mathcal{A}$, is given by

$$\mathcal{A} = \iint_D y(P-Q)\,dxdy + \mathcal{C}[P] - \mathcal{C}[Q]. \tag{180}$$

Setting $P = Q + q$, and substituting (179) for C, yields

$$\mathcal{A} = \iint_D \Big\{yq - \int_Q^{Q+q} Y(\tilde{q})\,d\tilde{q}\Big\}\,dxdy. \tag{181}$$

Finally, since $y = Y(Q)$, we may write

$$\mathcal{A} \equiv \iint_D A\,dxdy = \iint_D \Big\{-\int_0^{q} [Y(Q+\tilde{q}) - Y(Q)]\,d\tilde{q}\Big\}\,dxdy \tag{182}$$

(Killworth & McIntyre 1985). Note the similarity between the pseudomomentum (182) and the expression in (98) for the APE. As a consequence, (182) has the same geometrical interpretation as the APE (see the sketch in §3.4).

If dQ/dy (and thus dY/dQ) is sign-definite, then so is $\mathcal{A}$. In particular if $dY/dQ \neq 0$ and

$$0 < c_1 \le \left|\frac{dY}{dQ}\right| \le c_2 < \infty, \tag{183}$$

then

$$\frac{1}{2}c_1 q^2 \le |A| \le \frac{1}{2}c_2 q^2. \tag{184}$$

This is the convexity condition for this problem. We then have normed stability under the enstrophy norm. In particular, if we define our norm according to

$$\|q\|^2 = \iint_D \frac{1}{2}q^2\,dxdy, \tag{185}$$

then we have

$$\|q(t)\|^2 \leq \frac{1}{c_1}\mathcal{A}(t) = \frac{1}{c_1}\mathcal{A}(0) \leq \frac{c_2}{c_1}\|q(0)\|^2, \tag{186}$$

which proves normed stability.

As with the previous stability criteria, this stability criterion applies to arbitrarily large disturbances. It is the finite-amplitude version of the Rayleigh-Kuo theorem (Shepherd 1988*b*). The same procedure applied to the quasi-geostrophic equations yields a finite-amplitude Charney-Stern theorem (Shepherd 1988*a*, 1989).

In §3.6 and §4.4, pseudoenergy-based finite-amplitude stability theorems were used to obtain rigorous upper bounds on the nonlinear saturation of instabilities. The same procedure is of course possible with the pseudomomentum. For a general discussion and applications to parallel flows on the barotropic β-plane, see Shepherd (1988*b*). Further applications are provided in Shepherd (1988*a*, 1989, 1991).

5.3 Wave, mean-flow interaction

In this section we shed some light on why $\mathcal{A}$ is called the pseudomomentum. Still considering the barotropic vorticity equation, if we take the x-average of the zonal momentum equation we get

$$\frac{\partial \overline{u}}{\partial t} = -\overline{\frac{\partial (u^2)}{\partial x}} - \overline{\frac{\partial (uv)}{\partial y}} + f\overline{v} - \overline{\frac{\partial p}{\partial x}}. \tag{187}$$

The first and last two terms on the right-hand side vanish due to the presumed periodicity in x, together with the fact that $v = -\psi_x$ so $\overline{v} = 0$. This leaves

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial \overline{(u'v')}}{\partial y}, \tag{188}$$

where the primes indicate departures from the x-average flow. Using the fact that the flow is non-divergent, we can rewrite the previous equation as

$$\frac{\partial \overline{u}}{\partial t} = -\overline{v'\left(\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}\right)} + \overline{\frac{\partial}{\partial x}\left[\frac{1}{2}\left(u'^2 - v'^2\right)\right]}. \tag{189}$$

The second term on the right-hand side vanishes under the zonal average, while the first term represents the meridional flux of potential vorticity, q', hence

$$\frac{\partial \overline{u}}{\partial t} = \overline{v'q'}. \tag{190}$$

On the other hand, from the linearized potential vorticity equation

$$q'_t + Uq'_x + v'Q_y = 0 \tag{191}$$

we get

$$v' = -\frac{1}{Q_y}\left(q'_t + Uq'_x\right). \tag{192}$$

Substituting this expression for v' into (190) then leads to

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial}{\partial t}\Big(\frac{1}{2}\frac{\overline{q'^2}}{Q_y}\Big). \tag{193}$$

This is the well-known relation of Taylor (1915), describing how disturbance growth or decay induces mean-flow changes. But note that the small-amplitude limit of the pseudomomentum density (182) is

$$A \approx -\frac{1}{2}\frac{q'^2}{Q_y}. \tag{194}$$

Combining this with the previous equation then yields the small-amplitude relation

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial \overline{A}}{\partial t}, \tag{195}$$

which justifies the interpretation of $\mathcal{A}$ as a pseudomomentum. (The prefix "pseudo" has led to some confusion. However, to discuss the "momentum" of waves has historically been a source of profound confusion! For background on this issue, as well as a defense of the current nomenclature, the reader is referred to the spirited article of McIntyre (1981).)

In the continuously stratified quasi-geostrophic case, (190) generalizes to (see e.g. Pedlosky 1987, §6.14)

$$\mathcal{L}\Big(\frac{\partial \overline{u}}{\partial t}\Big) = \frac{\partial^2}{\partial y^2}\overline{(v'q')}, \tag{196}$$

where $\mathcal{L}$ is the linear elliptic operator

$$\mathcal{L} = \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s}{S}\frac{\partial}{\partial z}. \tag{197}$$

The pseudomomentum conservation relation may be written in local form, *including non-conservative effects*, as

$$\frac{\partial A}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{F} = D, \tag{198}$$

where D represents the non-conservative effects and $-\mathbf{F}$ is the so-called Eliassen-Palm flux (Andrews & McIntyre 1976), satisfying

$$\boldsymbol{\nabla}\cdot\overline{\mathbf{F}} = -\overline{v'q'}. \tag{199}$$

(The minus sign in the definition of the E-P flux is for historical reasons: the introduction of the E-P flux predated its understanding in terms of pseudomomentum.) From these relations we get the following equation for the mean-flow tendency:

$$\mathcal{L}\Big(\frac{\partial \overline{u}}{\partial t}\Big) = \frac{\partial^2}{\partial y^2}\overline{(v'q')} = -\frac{\partial^2}{\partial y^2}\big(\boldsymbol{\nabla}\cdot\overline{\mathbf{F}}\big) = \frac{\partial^2}{\partial y^2}\Big(\frac{\partial \overline{A}}{\partial t} - \overline{D}\Big). \tag{200}$$

Relation (200) generalizes (195) in two distinct ways: first, by including non-conservative effects; and second, by extending the relation to quasi-geostrophic flow. Integrating (200) in time over some finite time interval then gives the following expression for the net change in the zonal flow, $\Delta\overline{u}$:

$$\Delta\overline{u} = \mathcal{L}^{-1}\Big\{\frac{\partial^2}{\partial y^2}\Big(\Delta\overline{A} - \int \overline{D}\,dt\Big)\Big\}. \tag{201}$$

That is, to have a change in $\overline{u}$, we need either transience in the wave activity, $\Delta\overline{A} \neq 0$, or wave-activity dissipation, $\overline{D} \neq 0$; this is the "non-acceleration theorem" (Andrews & McIntyre 1976). The insight provided by this theoretical framework has recently led to profound advances in our understanding of some classical questions in large-scale atmospheric dynamics, including the maintenance of the westerlies (see the discussion in Shepherd 1992*b*).

The beauty of the Hamiltonian framework is that it provides insight into which aspects of a particular derivation may be generalizable to other systems. For example, the wave, mean-flow interaction theory exemplified by the relation (200) is clearly generalizable through the unifying concept of pseudomomentum (e.g. Scinocca & Shepherd 1992; Kushner & Shepherd 1993).

5.4 Wave action

There is a classical literature in fluid mechanics that is relevant to wave propagation in inhomogeneous, moving media. For WKB conditions — namely, a nearly monochromatic wave packet propagating in a slowly varying background state — there is a conservation law for the so-called "wave action" (Whitham 1965; Bretherton & Garrett 1968). The wave action is given by $E'/\hat{\omega}$, where E' is the wave energy (as measured in the local frame of reference, moving with the mean flow) and $\hat{\omega}$ is the intrinsic frequency of the waves (i.e. the frequency in the local frame of reference). In the case of the barotropic vorticity equation with a zonal basic state, for example,

$$E' = \frac{1}{2}\overline{|\nabla\psi'|^2} \qquad \text{and} \qquad \hat{\omega} = -\frac{kQ_y}{k^2+l^2}, \tag{202}$$

where k and l are the x and y wavenumbers, respectively, Q_y is the basic-state potential-vorticity gradient, and the overbar now represents an average over the phase of the waves. Thus the wave action for Rossby waves is given by

$$\frac{E'}{\hat{\omega}} = -\frac{1}{2k}\frac{(k^2+l^2)\overline{|\nabla\psi'|^2}}{Q_y} = -\frac{1}{2k}\frac{\overline{q'^2}}{Q_y}. \tag{203}$$

Referring to (194), we conclude that the wave action is the pseudomomentum divided by the zonal wavenumber,

$$\frac{E'}{\hat{\omega}} = \frac{A}{k}. \tag{204}$$

Of course, wave action is a local concept which may be defined even when there is no global symmetry in the problem (provided the WKB conditions are satisfied). However, when the basic state *has* a zonal symmetry, the pseudomomentum may be defined and is related to the wave action in the above fashion; the factor of k is then irrelevant since it is constant. Under such conditions, the pseudomomentum may be regarded as a generalization of wave action insofar as it is not restricted to WKB (slowly varying) conditions, neither is it restricted to small amplitude. This connection has already been made by Andrews & McIntyre (1978) within the context of Generalized Lagrangian Mean theory; the present treatment illustrates how it holds for the Eulerian formulation of fluid dynamics.

5.5 Instabilities

Analysis of normal-mode instabilities is often facilitated by the use of quadratic invariants such as pseudomomentum and pseudoenergy. This is because for a normal mode,

$$\mathcal{A} = \mathcal{A}_0 e^{2\sigma t} \tag{205}$$

where $\mathcal{A}_0 \equiv \mathcal{A}(t=0)$ and σ is the real part of the growth rate. However, $\mathcal{A}$ is conserved in time, which implies that $\sigma\mathcal{A} = 0$. Therefore, we conclude that growing or decaying normal modes (with $\sigma \neq 0$) must have $\mathcal{A} = 0$. [In fact, many of the well-known derivations of linear stability criteria involve the implicit use of this relation $\sigma\mathcal{A} = 0$: an example is Pedlosky (1987, Eq.(7.4.22))].

The constraint $\mathcal{A} = 0$ on normal-mode instabilities means that such instabilities consist of regions of positive and negative A. This is a generalization of the notion of positive and negative energy modes discussed in Morrison's lectures. It is clear from the Hamiltonian perspective that one may speak of positive and negative pseudoenergy, or positive and negative pseudomomentum, or even some combination of the two, depending on which invariant quantity is most appropriate for the problem at hand.

This concept is most useful when the wave-activity invariant being considered is sign-definite in certain parts of the flow, and can be associated (in an appropriate limiting sense) with certain wave modes. Typically in the short-wave limit these modes decouple and are neutral.

As an example, consider baroclinic instability in the continuously stratified quasi-geostrophic model, with $z_0 \leq z \leq z_1$. In this case the small-amplitude expansion of the pseudomomentum gives (Shepherd 1989)

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \tag{206}$$

where

$$\mathcal{A}_1 = -\iiint_D \frac{\rho_s}{2} \frac{q^2}{dQ/dy}\, dxdydz, \tag{207}$$

$$\mathcal{A}_2 = \iint \frac{1}{2} \frac{\lambda_1^2}{d\Lambda_1/dy}\, dxdy\bigg|_{z=z_1}, \qquad \mathcal{A}_3 = -\iint \frac{1}{2} \frac{\lambda_0^2}{d\Lambda_0/dy}\, dxdy\bigg|_{z=z_0}. \tag{208}$$

Here Q and q are the basic-state and disturbance potential vorticity fields, while $\Lambda_i = \frac{\rho_s}{S}\Psi_z\big|_{z=z_i}$ and $\lambda_i = \frac{\rho_s}{S}\psi_z\big|_{z=z_i}$ where Ψ and ψ are the basic-state and disturbance stream function fields. All known (inviscid) quasi-geostrophic baroclinic instabilities may be understood within this framework. In the case of the Eady model, we have

$$\frac{dQ}{dy} = 0, \quad q = 0, \quad \text{and} \quad \frac{d\Lambda_i}{dy} < 0. \tag{209}$$

Therefore, in this model instability is possible with $\mathcal{A}_1 = 0$, $\mathcal{A}_2 < 0$, and $\mathcal{A}_3 > 0$. In the case of the Charney model there is no upper lid so the contribution to $\mathcal{A}_2$ vanishes, while

$$\frac{dQ}{dy} > 0 \quad \text{and} \quad \frac{d\Lambda_0}{dy} < 0. \tag{210}$$

Thus in this case $\mathcal{A}_1 < 0$, $\mathcal{A}_2 = 0$, and $\mathcal{A}_3 > 0$. For internal baroclinic instability (like in the Phillips model), $\mathcal{A}_2 = 0$ and $\mathcal{A}_3 = 0$ so we must have $\mathcal{A}_1 = 0$, but characteristically

$$\frac{dQ}{dy} > 0 \quad \text{for} \quad z > z_c \quad \text{and} \quad \frac{dQ}{dy} < 0 \quad \text{for} \quad z < z_c \tag{211}$$

for some z_c, so $\mathcal{A}_1$ consists of a negative-A mode above a positive-A mode.

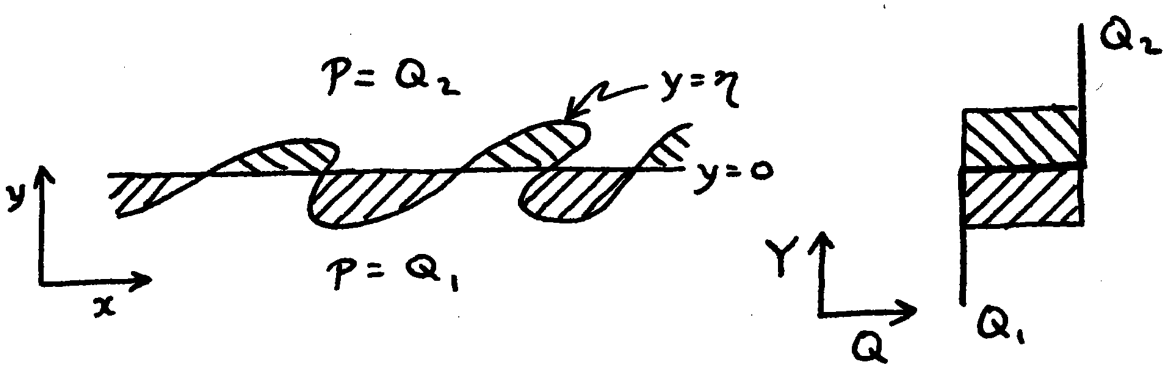
A very important feature of these wave-activity invariants is that their finite-amplitude forms are meaningful even for discontinuous basic-state profiles. Indeed, the understanding of instabilities in terms of interacting modes is clearest when the modes are spatially localized on material interfaces. For example, consider the barotropic system with a basic state

$$Q(y) = \begin{cases} Q_2, & y > 0 \\ Q_1, & y < 0 \end{cases} \tag{212}$$

where $Q_1 < Q_2$. We can study the stability of this profile by looking at the regions where $A \neq 0$. In this case the pseudomomentum is given by

$$A = -\int_0^q \left[Y(Q + \tilde{q}) - Y(Q)\right] d\tilde{q}. \tag{213}$$

Note that $A = 0$ except in the hatched regions (see figure below).



It turns out (see Shepherd 1988*b*, Appendix A) that

$$\mathcal{A} = \iint_D A\, dxdy = -\frac{1}{2}(Q_2 - Q_1) \oint \eta^2\, dx, \tag{214}$$

where η is the meridional displacement of the material contour where the vorticity jump occurs. Evidently in this case $\mathcal{A} < 0$, and the basic state (212) is stable. The above formula can be generalized for N contours (denoted by C_i) as follows:

$$\mathcal{A} = \iint_D A\, dxdy = -\frac{1}{2}\sum_{i=1}^{N} \oint_{C_i} \eta^2\, dx. \tag{215}$$

So we see that the pseudomomentum resides in each contour, and has a sign opposite to that of the vorticity jump. This is in contrast to the pseudoenergy, which is not so localized.

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