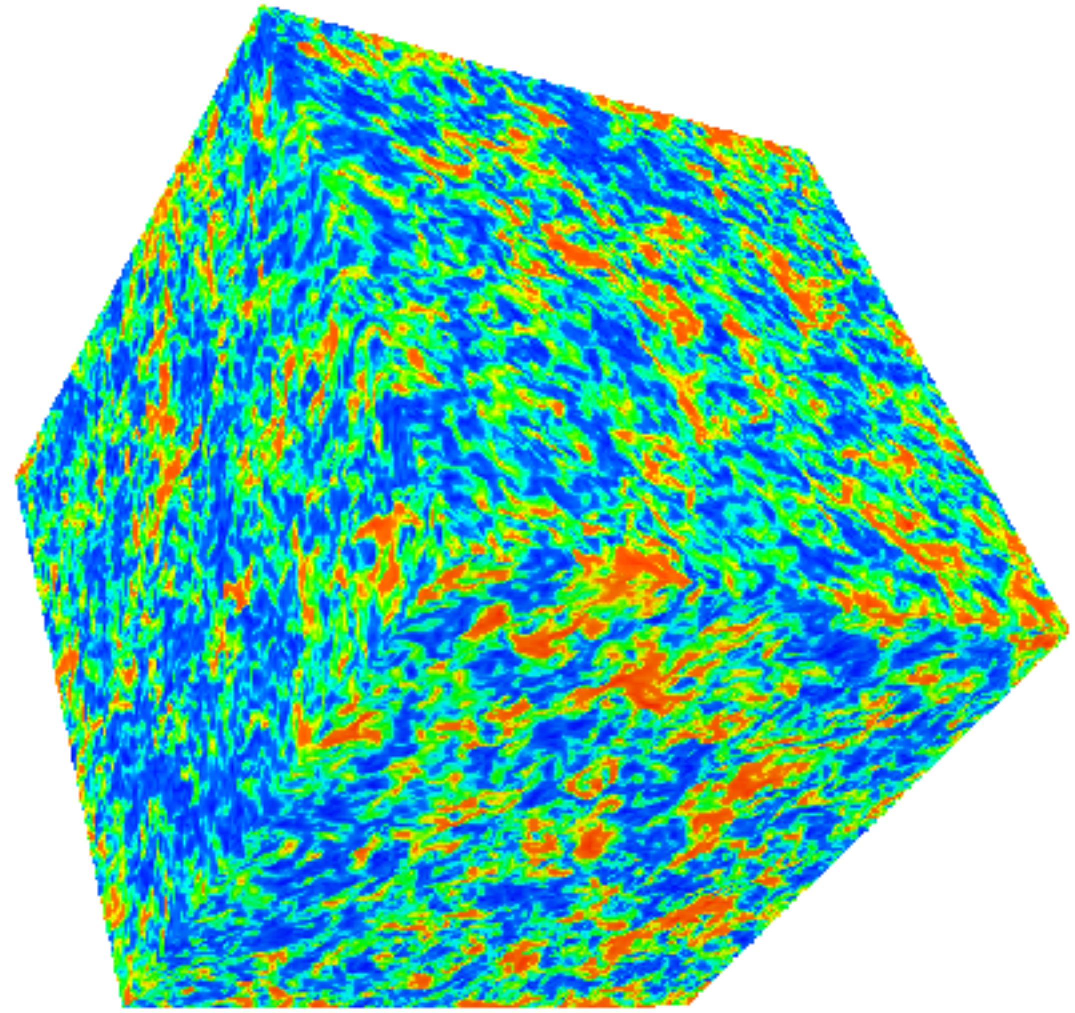
The zeroth law of turbulence and its discontents

$C_D \approx 0.4$



If in an experiment on turbulent flow all control parameters are fixed, except for the viscosity, which is lowered as much as possible, the energy dissipation per unit mass approaches a nonzero limit.

Turbulence - the Legacy of A.N. Kolmogorov
Uriel Frisch (1996)

An example of the zeroth law of turbulence

"If my views be correct, a fall of 800 feet will generate one degree of heat, and the temperature of the river Niagara will be raised about one fifth of a degree by its fall of 160 feet." (Joule 1845)

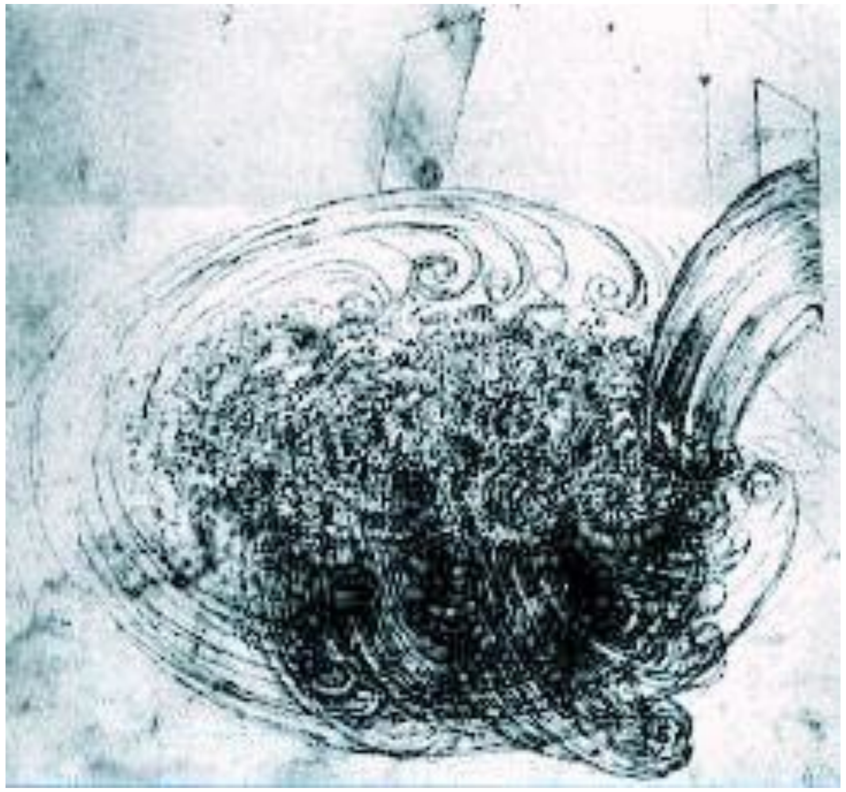
$$\Delta T = \frac{gH}{c_p} = \frac{10 \times 48}{4200}\mathrm{K} = 0.11\mathrm{K}$$

$$c_p = 4200\mathrm{JK}^{-1}\mathrm{kg}^{-1}$$

(For seawater c_p is close to 4000MKS)

This dissipative process requires molecular viscosity. But the coefficient of viscosity ν is very small and Joule takes it as given, or obvious, that the exact value of ν unimportant.

Wikimedia



$$\nu_{\mathrm{water}} \approx 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$$

$$R = \frac{\sqrt{gH}H}{\nu_{\mathrm{water}}} \sim 10^{10}$$

That's a great story about Joule's honeymoon, but is it true?
He did the calculation, but did he measure the temperature rise?

That's a great story about Joule's honeymoon, but is it true? He did the calculation, but did he measure the temperature rise?

- The temperature increase, *0.1* degree, is smaller than the expected variation of air temperature over *H=50m*. (For example, the average tropospheric lapse rate is 6.6 degrees per kilometer.)
- But is the water in thermal equilibrium with atmosphere?
- What about air drag on falling drops, and evaporative cooling of spray?
- The temperature increase is buried in a lot of noise and the story is probably apocryphal…..

(Craig Bohren, American Journal of Physics)

Another example

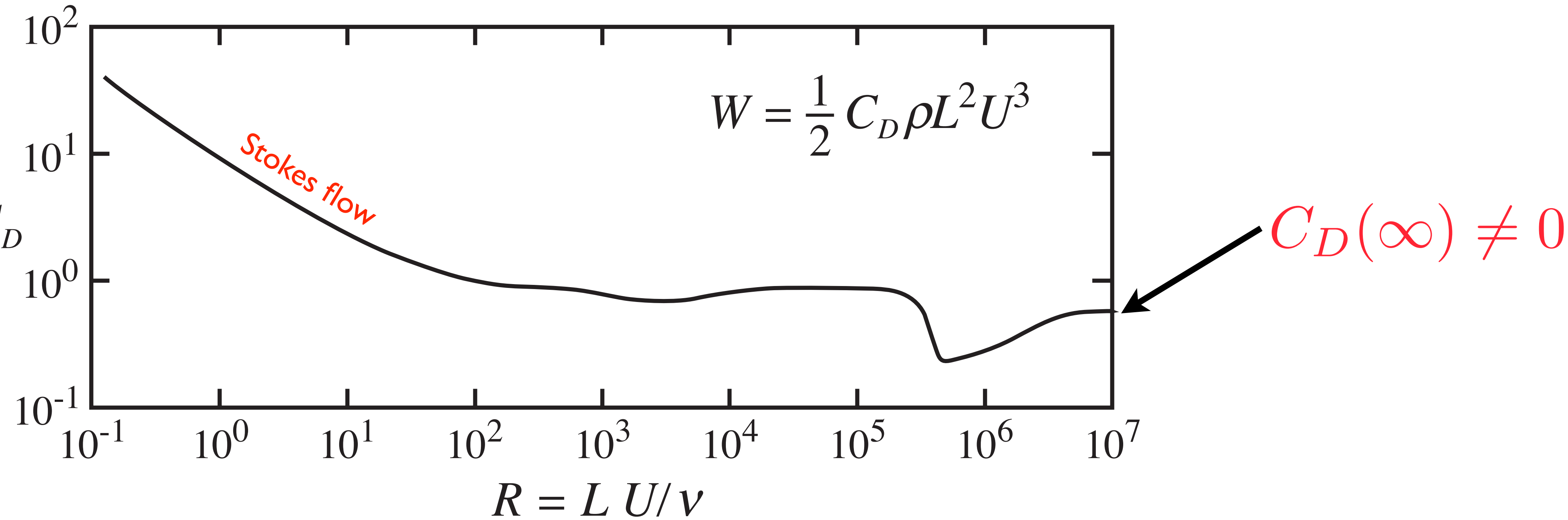
$\nu_{\text{air}} \approx 10^{-5}\mathrm{m\,s^{-1}}$

$R \sim 10^6$

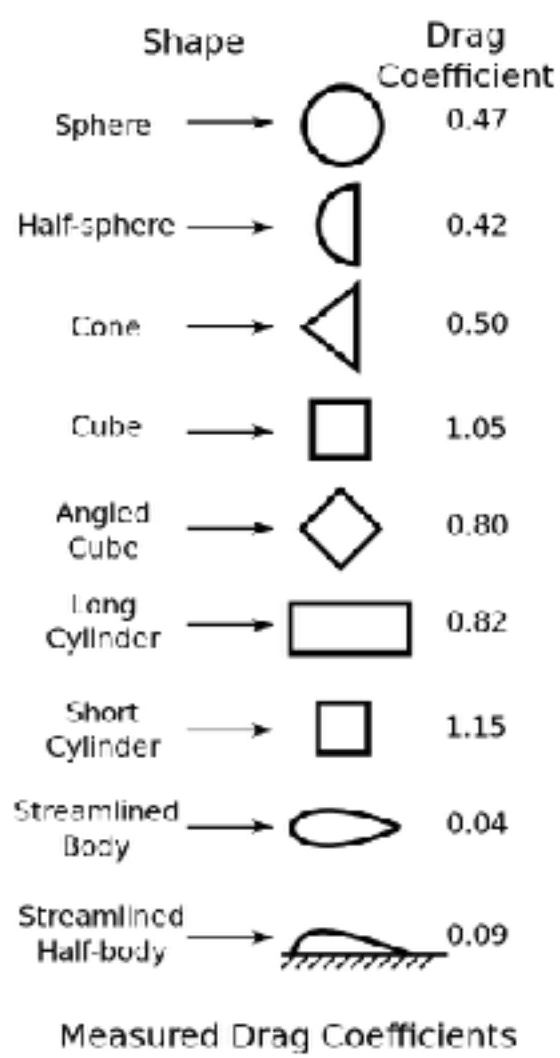
$$\text{Drag} = \frac{1}{2} C_D(R) \rho A U^2 \qquad \text{with} \qquad R = \frac{UL}{\nu}$$

and $\lim_{R\to\infty} C_D(R)$ is non-zero

“Turbulence” Uriel Frisch (1996)



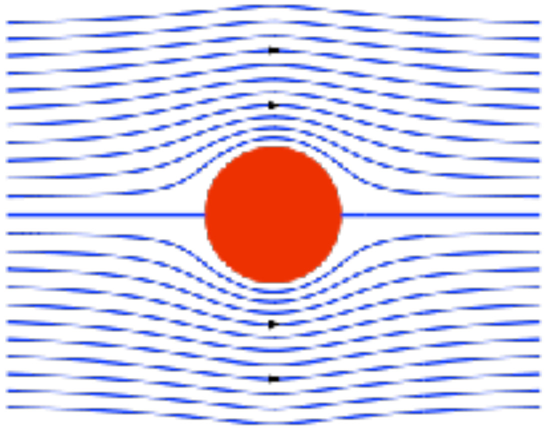
From Wikipedia



The drag coefficient means the limit at infinite Reynolds number.

There is no drag without viscosity, but ultimately the value of ν is irrelevant.
$\nu \to 0$ is a singular limit.
This is the zeroth law of turbulence.

D’Alembert’s paradox



“The theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers”

D’Alembert

“Fluid mechanics is divided between hydraulic engineers who observe phenomena which cannot be explained and mathematicians who explain phenomena which cannot be observed.”

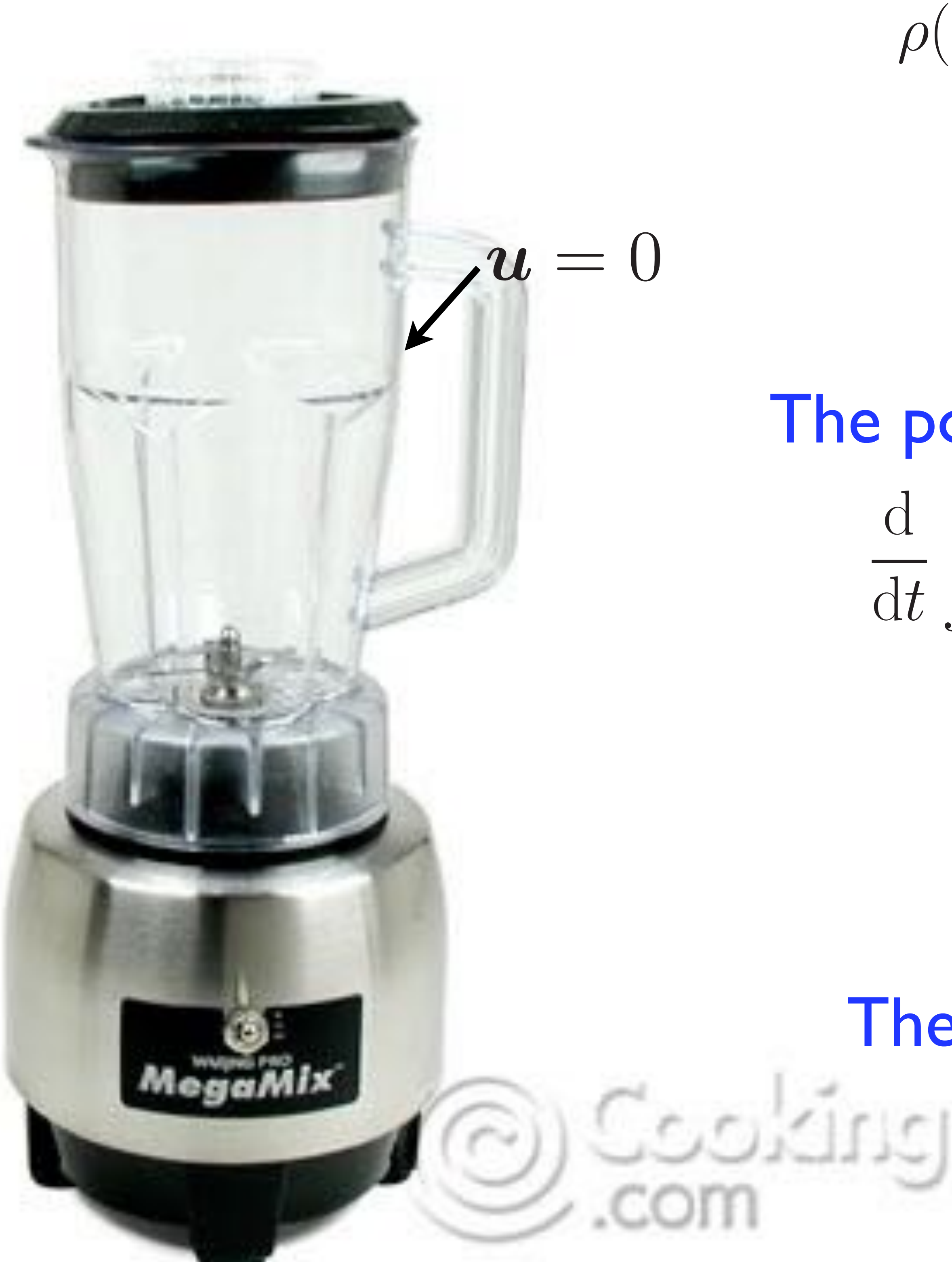
Cyril Hinselwood

The zeroth law of turbulence

If in an experiment on turbulent flow all control parameters are fixed, except for the viscosity, which is lowered as much as possible, the energy dissipation per unit mass approaches a nonzero limit.

“Turbulence - the Legacy of A.N. Kolmogorov”
Uriel Frisch (1996)

An enclosed incompressible fluid



$\boldsymbol{u} = 0$

$$\rho(\boldsymbol{u}_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}) + \boldsymbol{\nabla} p = \mu\nabla^2\boldsymbol{u} + \boldsymbol{f}$$

and $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$

The power integral is:

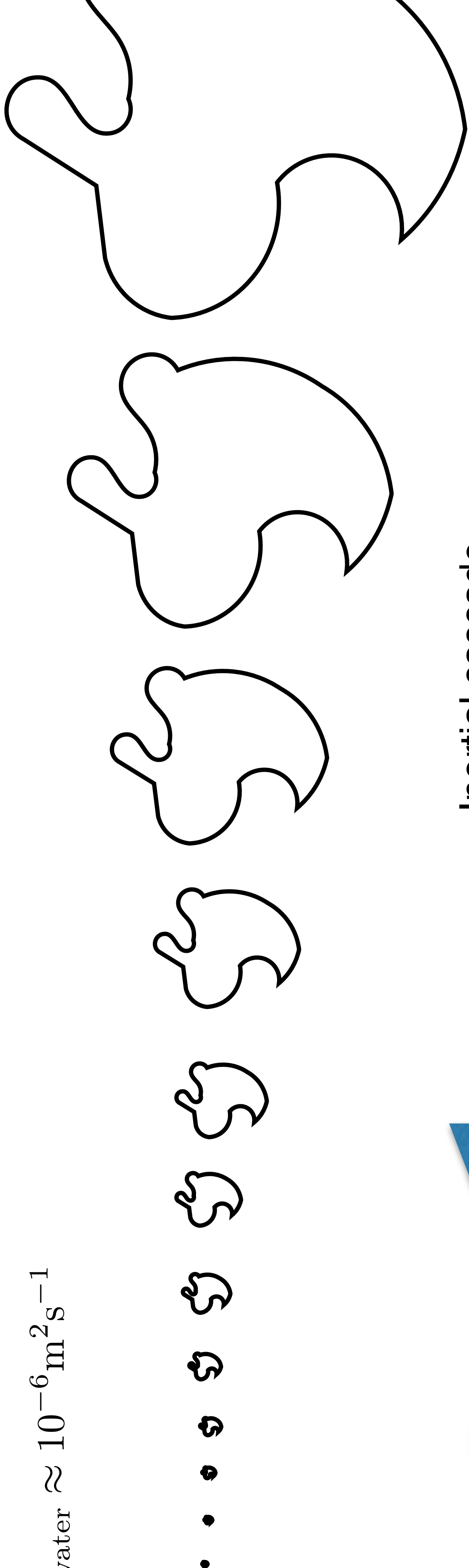
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \tfrac{1}{2}|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} = \int_V \boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x} - \nu\int_V |\boldsymbol{\omega}|^2\,\mathrm{d}\boldsymbol{x}$$

with the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$

The zeroth law is: $\lim\limits_{\nu\to 0}\nu\int_V |\boldsymbol{\omega}|^2\,\mathrm{d}\boldsymbol{x} \neq 0$

Turbulence is a singular limit
(a "dissipative anomaly").

3D turbulent energy cascade



Injection scale L

Inertial cascade

$$\varepsilon \sim \frac{U_{\mathrm{RMS}}^3}{L}$$

$$\dim \varepsilon = \frac{\mathrm{W}}{\mathrm{kg}} = \frac{L^2}{T^3}$$

Viscous dissipation scale

$$\ell_{\mathrm{K}} = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

A.K.A. the
Kolmogorov scale

$\nu_{\mathrm{water}} \approx 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$

Big whorls have little whorls
Which feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity.
— L. F. Richardson

Aside: why turbulence in a coffee cup is “stronger” than turbulence in the ocean.

$$\ell_{\mathrm{K}}^{\mathrm{coffee}} = \left(\frac{10^{-18}}{0.1}\right)^{1/4} = 6 \times 10^{-5}\,\mathrm{m}$$

$$\ell_{\mathrm{K}}^{\mathrm{ocean}} = \left(\frac{10^{-18}}{10^{-9}}\right)^{1/4} = 6 \times 10^{-3}\,\mathrm{m}$$

(Assuming your wrist outputs 0.01W to stir 0.1kg of coffee.)

Two examples of flows that can never be turbulent according to the zeroth law.

Horizontal Convection

and

Two-dimensional “turbulence”

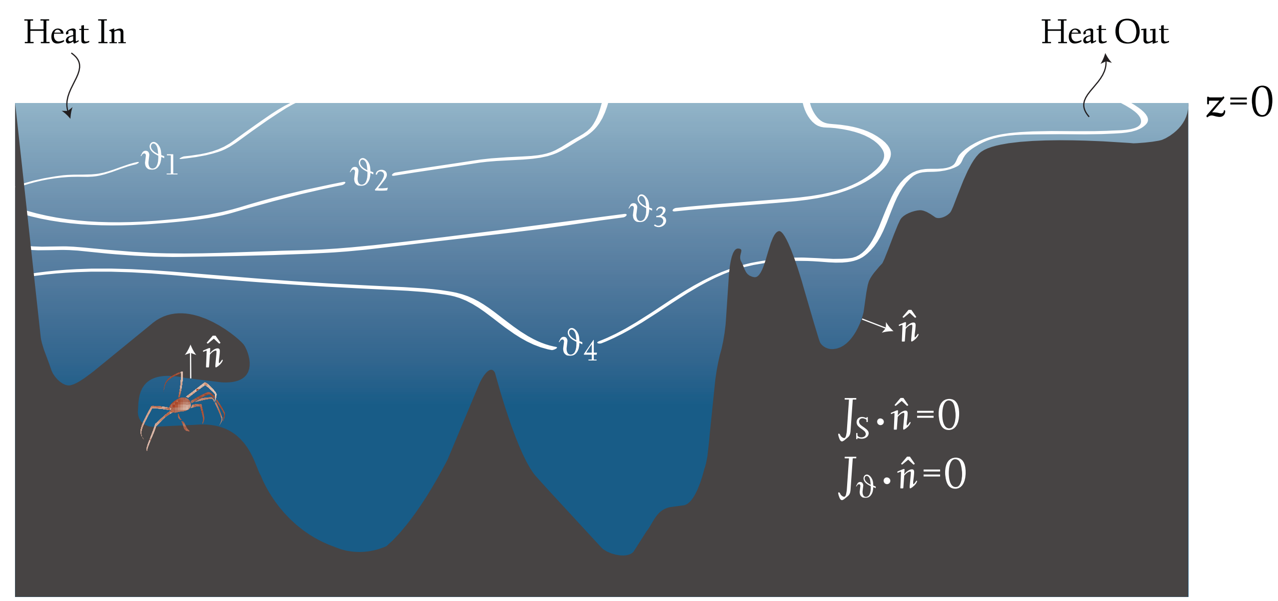
If in an experiment on turbulent flow all control parameters are fixed, except for the viscosity, which is lowered as much as possible, the energy dissipation per unit mass approaches a nonzero limit.

Turbulence - the Legacy of A.N. Kolmogorov
Uriel Frisch (1996)

Motivation for HC

The ocean is heated and cooled (mostly) at the sea surface. HC is most idealized situation in which the implications of this observation can be studied.

"One of the striking features of the oceanic circulation is the smallness at the ocean surface of the regions where deep and bottom water is formed."
— Stommel (1962)



Abyssal recipes II:
energetics of tidal and wind mixing

Walter Munk[a,*], Carl Wunsch[b]

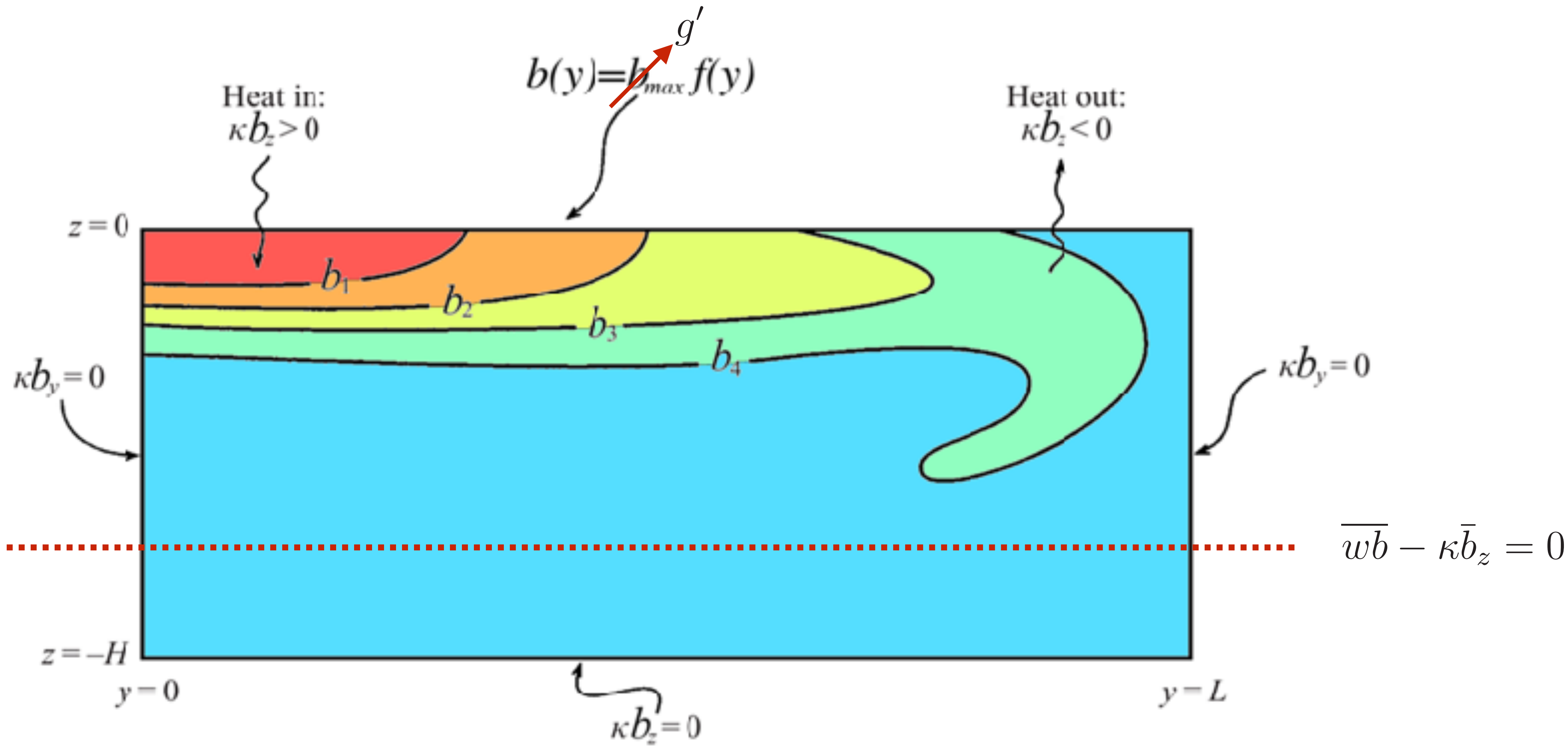
Without deep mixing, the ocean would turn, within a few thousand years, into a stagnant pool of cold salty water with equilibrium maintained locally by near-surface mixing and with very weak convectively driven surface-intensified circulation. (This result follows from Sandström's theorem for a fluid heated and cooled at the surface.) In this context we revisit the 1966

And the mysterious
Sandstrom Theorem

Horizontal Convection

Convection driven by heating and cooling at a single horizontal surface

Paparella & Young, (2002) JFM



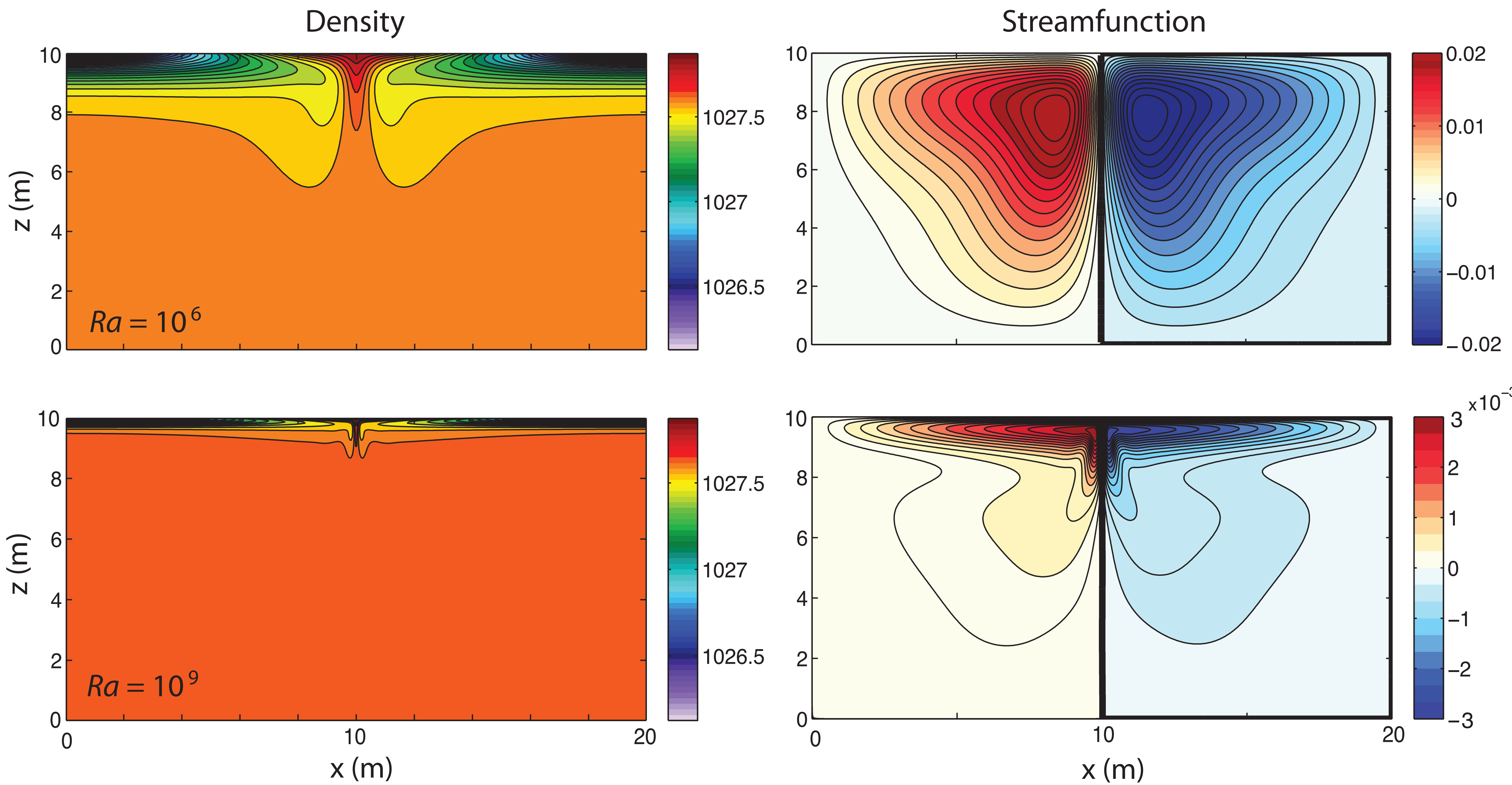
The definition of “buoyancy”: $\rho = \rho_0 \left(1 - g^{-1} b(\boldsymbol{x}, t)\right)$

Conventional notation: $b = g\alpha \left(T - T_0\right)$

Note an important difference from RBC:
the zero-flux constraint.

$$\mathrm{Ra} = \frac{\Delta b\, L^3}{\nu\kappa}$$

2D numerical HC solutions. Fluid sinks in the center. The circulation is weaker and the thermocline thinner at higher Rayleigh number.



Vallis (2017)

At very high Ra the box fills with the densest available water. (Unlike RBC.)

Without deep mixing, the ocean would turn, within a few thousand years, into a stagnant pool of cold salty water with equilibrium maintained locally by near-surface mixing and with very weak convectively driven surface-intensified circulation. (This result follows from Sandström's theorem for a fluid heated and cooled at the surface.) In this context we revisit the 1966

— Munk & Wunsch (1998)

HC in the laboratory
(the bottom is non-uniformly heated)

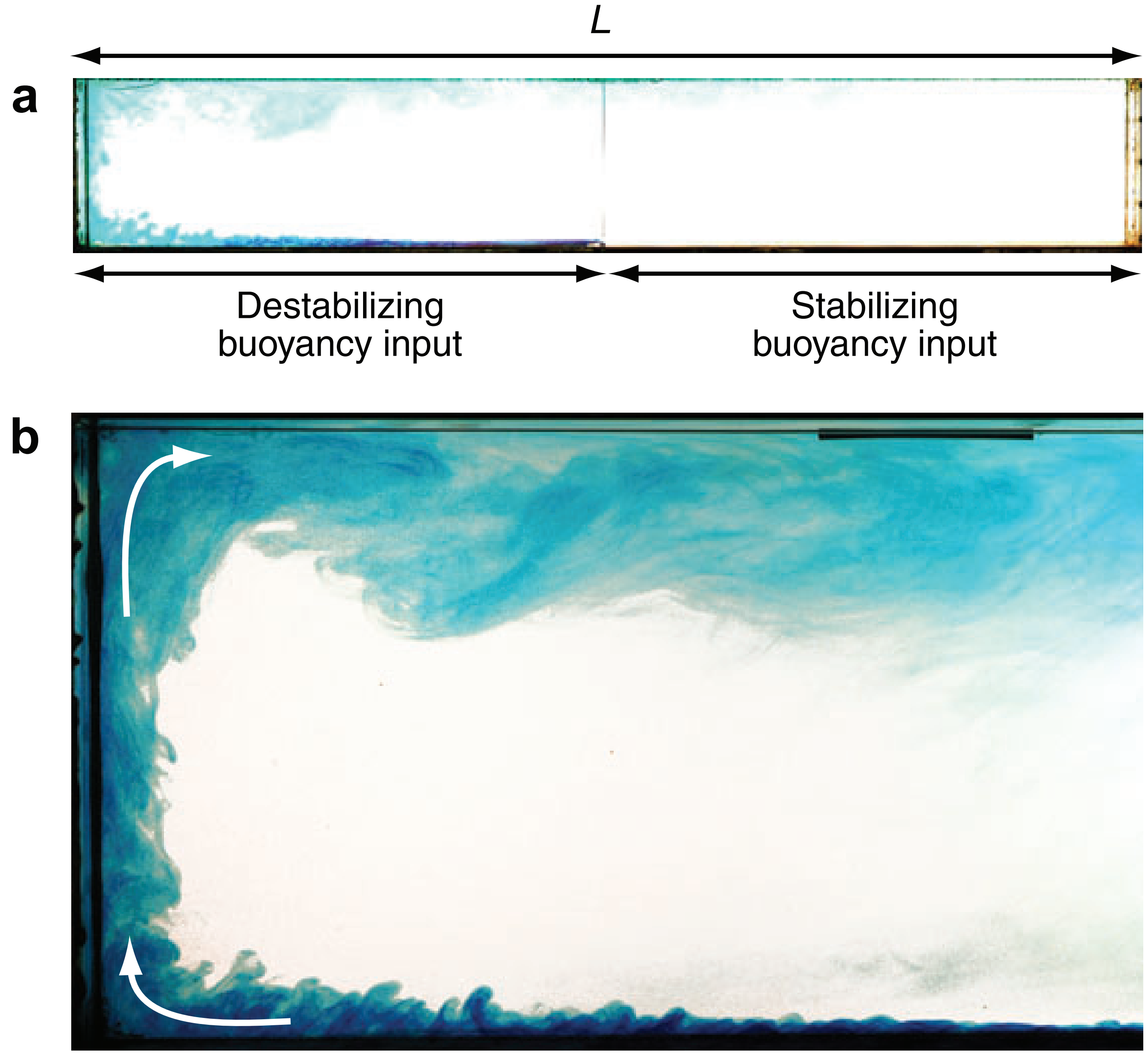
HC "explains" the asymmetry between sinking and rising regions in the ocean. (Unlike RBC.)

The box fills with the lightest available water. (Unlike RBC.)

The horizontal scale of the overturning is set by the box. (Unlike RBC.)

The critical Rayleigh number is zero. (Unlike RBC.)

Hughes & Griffiths, (2008) ARFM

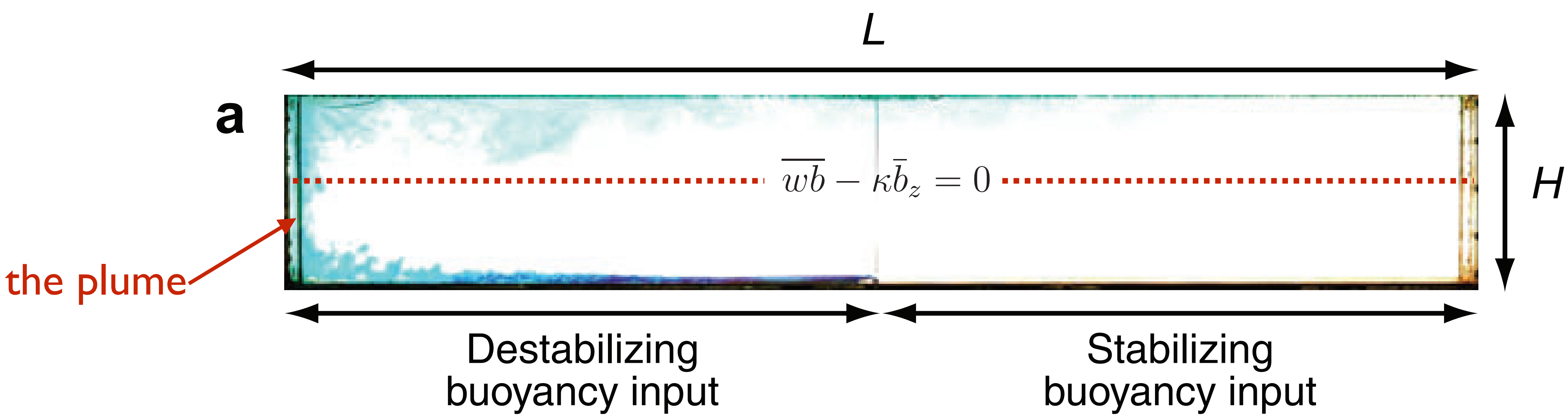


N.B. non-uniform buoyancy ↑ at the bottom!

Figure 1

Horizontal convection in a thermally equilibrated laboratory experiment subject to heating and cooling that depends on position along the base of the box (the rate of supply of specific buoyancy is $1.6 \times 10^{-6}\ \mathrm{m^3 s^{-3}}$ per unit width in the spanwise direction, using a uniform imposed heat flux at the left-hand end and an imposed boundary temperature of 16°C at the right-hand end) at a Rayleigh number $Ra_B = 2.2 \times 10^{14}$ and Prandtl number $Pr \approx 4$. Passive dye tracer is introduced into the bottom boundary layer halfway along the tank to visualize the circulation. Panel *a* shows the full box, whereas panel *b* is a close-up (approximately one-fourth the tank length) of the left-hand end, showing an asymmetric clockwise circulation extending through the depth of the box, with a convective mixed layer embedded in a stably stratified boundary layer on the base, an entraining plume against the vertical end wall, and eddies in the horizontal outflow from the plume. **Figure 1*b*** taken from Mullarney et al. 2004, courtesy of Cambridge University Press.

Why are the sinking regions in HC so small?



The short answer: upwards buoyancy flux in the plume is much more efficient than downwards diffusive buoyancy in the interior. But these fluxes cancel. Thus the convective plumes must cover a small area leaving most of the box for inefficient diffusive downwelling.

"One of the striking features of the oceanic circulation is the smallness at the ocean surface of the regions where deep and bottom water is formed."

— Stommel (1962)

Now show that HC cannot satisfy the zeroth law of turbulence.

If in an experiment on turbulent flow all control parameters are fixed, except for the viscosity, which is lowered as much as possible, the energy dissipation per unit mass approaches a nonzero limit.

Turbulence - the Legacy of A.N. Kolmogorov
Uriel Frisch (1996)

The equations of (Boussinesq) fluid motion.

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \hat{\boldsymbol{z}} \times f\boldsymbol{u} + \boldsymbol{\nabla} p = b\hat{\mathbf{z}} + \nu\nabla^2\boldsymbol{u},$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = \kappa\nabla^2 b,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0.$$

Power integrals for HC.

$$\langle \boldsymbol{u} \cdot \text{momentum equation} \rangle \quad \Rightarrow \quad \nu \underbrace{\langle \|\nabla \boldsymbol{u}\|^2 \rangle}_{\stackrel{\text{def}}{=}\varepsilon} = \langle wb \rangle$$

$$\langle (-z) \times \text{buoyancy equation} \rangle \quad \Rightarrow \quad \langle wb \rangle = \kappa \frac{\bar{b}(0) - \bar{b}(-H)}{H}$$

Combine the power integrals.

$$\therefore \quad \varepsilon = \kappa \frac{\bar{b}(0) - \bar{b}(-H)}{H} \leq \frac{\kappa \Delta b(0)}{H},$$

$$\rightarrow 0 \qquad \text{as } \nu \rightarrow 0 \text{ with } \nu/\kappa \text{ fixed.}$$

$\therefore$ HC does not satisfy the zeroth law of turbulence.

Sandstrom's (1908) "theorem"

"A closed steady circulation can only be maintained if the heat source is below the closed source."

There are recent endorsements of the "theorem": Munk & Wunsch (1998), Huang (1999), Emmanuel (2001), Wunsch & Ferrari (2004) etc

Without deep mixing, the ocean would turn, within a few thousand years, into a stagnant pool of cold salty water with equilibrium maintained locally by near-surface mixing and with very weak convectively driven surface-intensified circulation. (This result follows from Sandström's theorem for a fluid heated and cooled at the surface.) In this context we revisit the 1966

Also many counterexamples: Jeffreys (1925), Rossby (1965), Mullarney et al. (2004), Wang & Huang (2005).

So why don't oceanographers abandon Sandstrom?

"HC isn't a vigorous flow"

"HC produces a thin thermocline — you need winds, tides and breaking IGWs to explain deep ocean stratification"

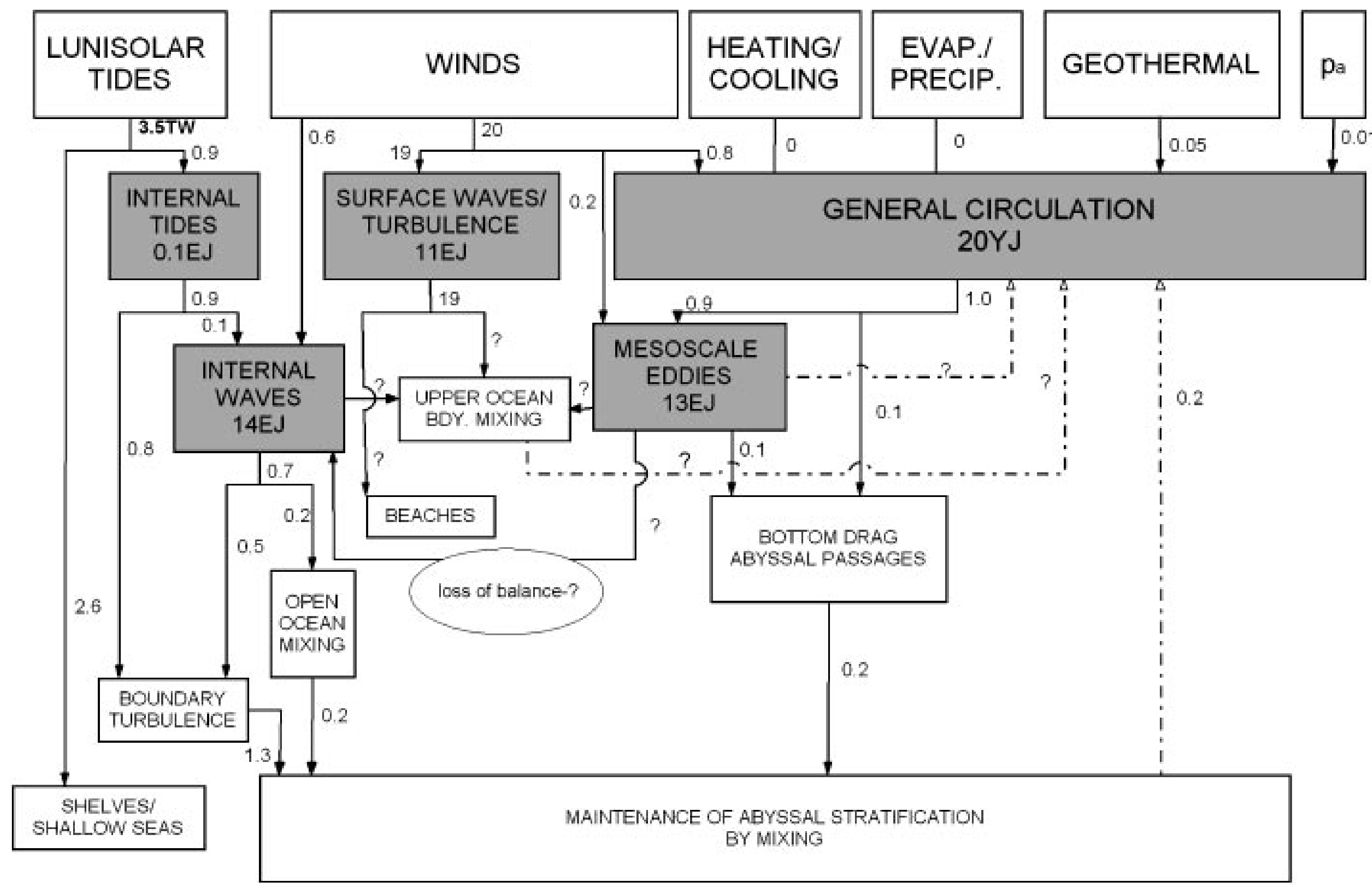
"Strict interpretation of the theorem is difficult" — Houghton (1977)

To turn the "theorem" into a theorem, and strictly interpret it, use this formula

$$\underbrace{\nu\langle|\boldsymbol{\nabla u}|^2\rangle}_{\varepsilon} = \kappa\,\frac{\bar{b}(0)-\bar{b}(-H)}{H}$$

But now the debate focusses on whether ε is important for ocean circulation. The majority position is at right.

302 WUNSCH ■ FERRARI



A ballpark estimate

$$\Delta T = 25\mathrm{K}\,, \qquad \alpha = 2\times10^{-4}\mathrm{K}^{-1}\,,$$
$$g = 10\mathrm{m\,s^{-2}}\,, \qquad \Rightarrow \qquad b_{\max} = 5\times10^{-2}\mathrm{m\,s^{-2}}$$

$$\begin{aligned}\varepsilon &\le \frac{\kappa\, b_{\max}}{h}\\ &= \frac{10^{-7}\times5\times10^{-2}}{5000}\\ &= 10^{-12}\,\mathrm{W\,kg^{-1}}\end{aligned}$$

Exercise (surprisingly easy)

Show that $$\langle P\,\boldsymbol{\nabla}\cdot\boldsymbol{U}\rangle \propto \kappa\,\frac{\bar{b}(0)-\bar{b}(-H)}{H} = \kappa\langle -z\nabla^2 b\rangle$$

$\boldsymbol{U}$ = the exact, non-Boussinesq compressible velocity

P the total pressure

$$\rho_t + \boldsymbol{\nabla}\cdot(\boldsymbol{U}\rho) = 0$$

You'll never doubt the Boussinesq approximation again.

Thus the HC energy source is given by Sandstorm's "piston formula": conversion of internal energy to mechanical energy.

But does this look like a laminar flow?

"We suggest that the "zeroth" law is too restrictive, since according to this strict definition even canonical Rayleigh-Benard convection would not be turbulent!"
-Scotti & White (2011)

Spiegel suggests that examples such as these should be called "thermalence".

"Horizontal convection can be interpreted in terms of a mechanical energy budget, but a detailed understanding has not emerged"
-Griffiths & Hughes (2007)

"These results explain why the convection is much stronger than might be inferred from previous emphasis on minor terms (the buoyancy flux viscous dissipation balance and the potential energy budget)"
-Gayen, Griffiths, Hughes and Saenz (2007)

$$\langle wb \rangle = \kappa \frac{\bar{b}(0) - \bar{b}(-H)}{H}$$

$$\langle wb \rangle = \nu \langle \|\nabla \boldsymbol{u}\|^2 \rangle$$

The second example of flows that do not satisfy the zeroth law is 2D turbulence.

The key feature of 2D turbulence is the robust conservation of energy,
and the transfer of energy to large scales.
(The inverse energy cascade, negative viscosity, anti-friction etc.)

What would Niagara Falls look like in Flatland?

The singular 3D limit results from vorticity production

$$\nabla \times \left[\rho(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla p = \mu \nabla^2 \boldsymbol{u} \right]$$

$$\boldsymbol{\omega}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}$$

vortex stretching

But in 2D

$$\boldsymbol{u} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} \qquad \boldsymbol{\omega} = (v_x - u_y)\hat{\boldsymbol{k}} \qquad \therefore \qquad \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = 0$$

So there is no turbulence in flatland — only flatulence.

The special structure of 2D fluid mechanics

$$u_x + v_y = 0 \qquad \Rightarrow \qquad u = -\psi_y \quad v = \psi_x$$

$$\zeta = v_x - u_y = \nabla^2 \psi$$

The curl of the momentum equation produces the 2D vorticity equation:

$$\zeta_t + \psi_x \zeta_y - \psi_y \zeta_x = \nu \nabla^2 \zeta$$

2D Conservation laws

Energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \tfrac{1}{2}|\boldsymbol{\nabla}\psi|^2\,\mathrm{d}x\mathrm{d}y = -\nu\iint \zeta^2\,\mathrm{d}x\mathrm{d}y$$

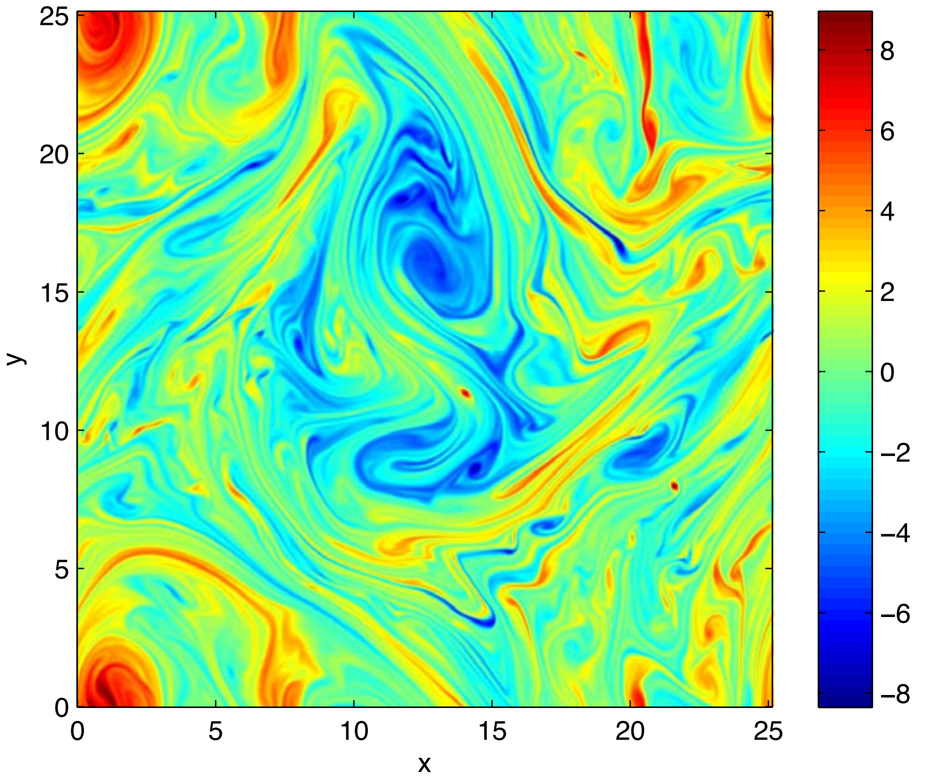
Enstrophy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \tfrac{1}{2}\zeta^2\,\mathrm{d}x\mathrm{d}y = -\nu\iint |\boldsymbol{\nabla}\zeta|^2\,\mathrm{d}x\mathrm{d}y$$

Take the limit $\nu \to 0$ and observe that enstrophy is bounded by its initial value. Therefore energy is conserved in the limit $\nu \to 0$.

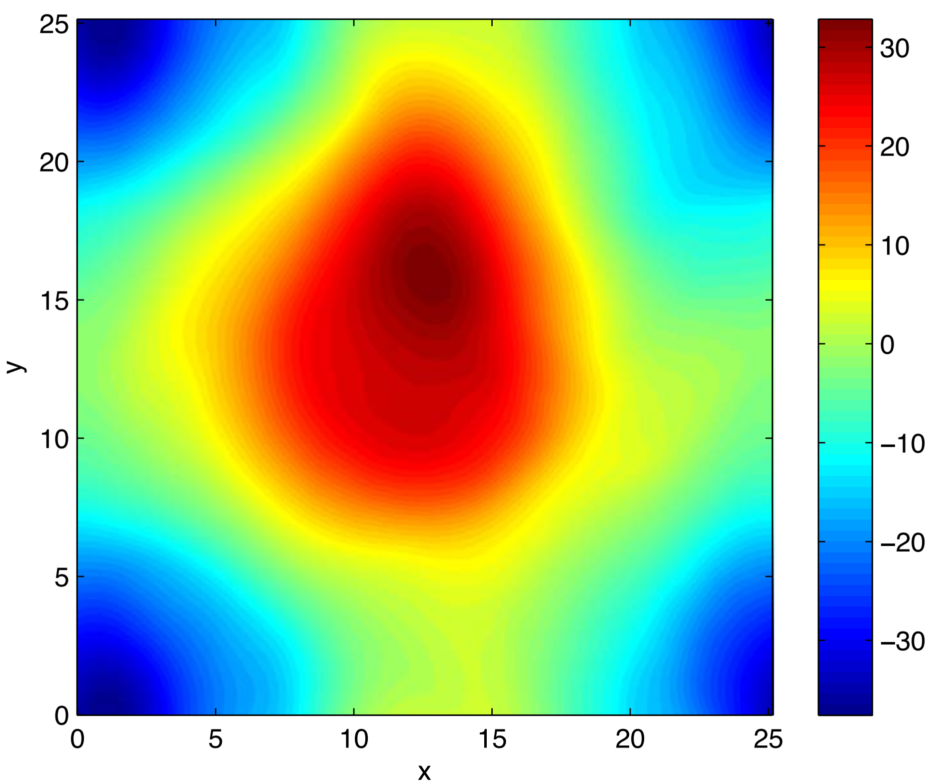
According to the zeroth law, there is no turbulence in flatland. Spiegel suggests that 2D turbulence should instead be called flatulence.

$$\zeta = \psi_{xx} + \psi_{yy}$$



☛Given the vorticity at $t=0$, calculate the streamfunction and the velocity: $\boldsymbol{u} = \hat{\boldsymbol{k}} \times \boldsymbol{\nabla}\psi$

☛Advect the vorticity for a time dt.



☛Calculate the new streamfunction.

☛Keep in mind that the vorticity cannot mix down to arbitrarily small scales: this would violate conservation of energy.

Consider an ideal fluid so that energy and enstrophy are both conserved. Then we can make a very plausible argument that energy is transferred to large scales.

$$\tfrac{1}{2}\langle u^2 + v^2 \rangle = \int_0^\infty E(k,t)\,\mathrm{d}k$$

$$\tfrac{1}{2}\langle \zeta^2 \rangle = \int_0^\infty k^2 E(k,t)\,\mathrm{d}k$$

$$E(k,t) = \text{the energy spectrum}$$

The OBF argument

☛The mean wavenumber of the energy spectrum is:

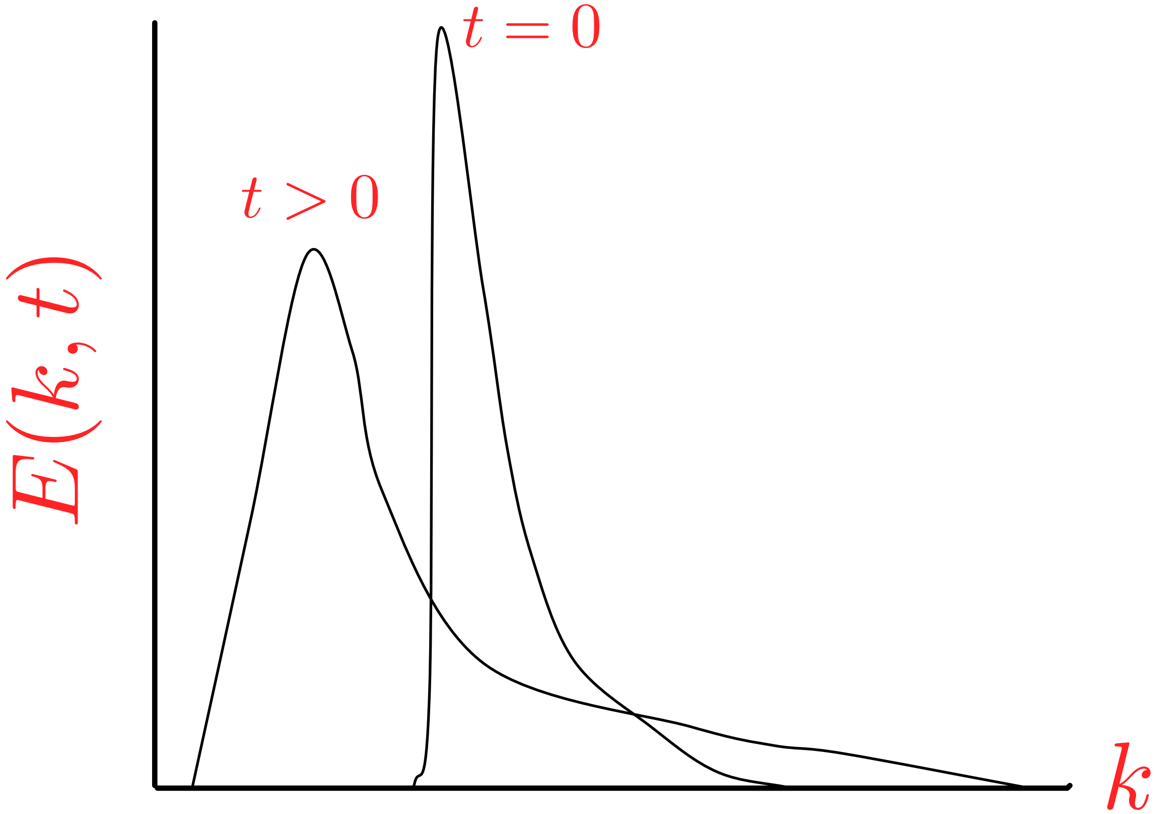
$$\bar{k}(t) = \frac{\int k E(k,t)\,\mathrm{d}k}{\int E(k,t)\,\mathrm{d}k}$$

☛The spectral width is:

$$\frac{\int (k-\bar{k})^2 E(k,t)\,\mathrm{d}k}{\int E(k,t)\,\mathrm{d}k} = \frac{\int k^2 E(k,t)\,\mathrm{d}k}{\int E(k,t)\,\mathrm{d}k} - \bar{k}^2$$

constant

☛If nonlinear interactions broaden an initially narrow spectrum, then the mean wavenumber must decrease.



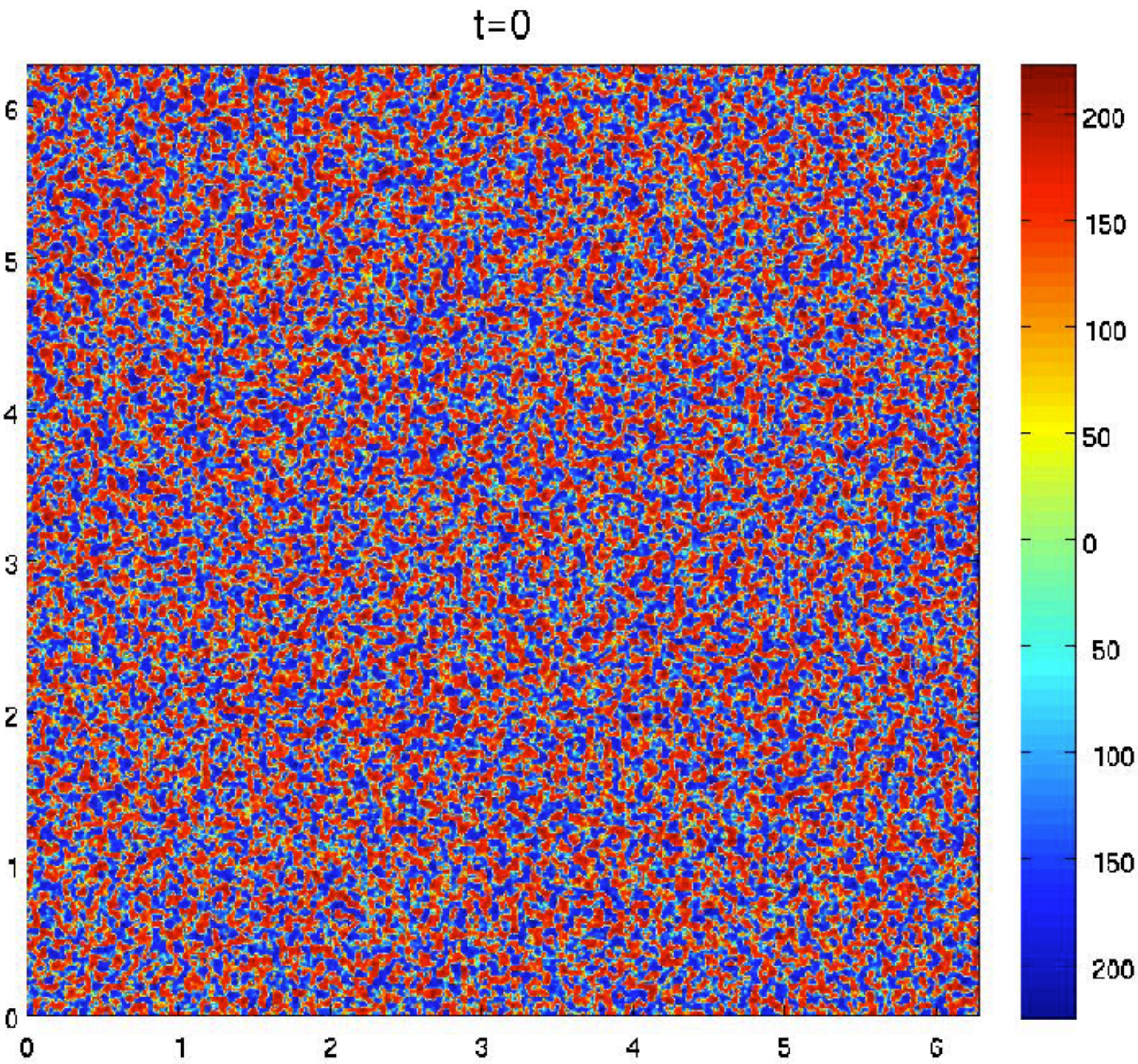
“The net tendency for the bulk of the energy to concentrate in the small wavenumbers means that fluid elements with similarly signed vorticity must tend to group together; in no other way is it possible for the scale of the velocity distribution to increase. We expect therefore that from the original motion there will gradually emerge a few strong isolated vortices and that vortices of the same sign will continue to tend to group together....

Onsager (1949) has arrived at a similar conclusion about the tendency for a small number of strong isolated vortices to form.”

Batchelor, *The Theory of Homogeneous Turbulence* (1953)

The Cray supercomputer came online in 1977 and flatland was settled by Benzi, Fornberg, McWilliams, Santangelo.....

Briscolini & Santangelo (1991), McWilliams (JFM 1984 &1990)
Benzi, Patarnello & Santangelo (1987)



$$\zeta(x, y, t)$$

- A random IC.

$$E(k,0) = \frac{C\,k}{1 + (k/6)^4}$$

- Emergence of a "vortex gas".
- Merger, straining & stripping.
- Energy is conserved throughout.
- Nonuniform color scale to show filaments.
- At intermediate times there is a scaling law

$$\varrho(t) \sim t^{-0.72}$$

- The final state is a dipole.

What did we just see?

$$\zeta_t + \psi_x \zeta_y - \psi_y \zeta_x = -3.125 \times 10^{-8} \nabla^4 \zeta$$

$$\zeta = \psi_{xx} + \psi_{yy}$$

$$E(k,0) = \frac{C\,k}{1 + (k/6)^4} \qquad \text{with} \qquad \langle u^2 + v^2 \rangle = 1$$

☛The flow organizes into a dilute vortex gas.

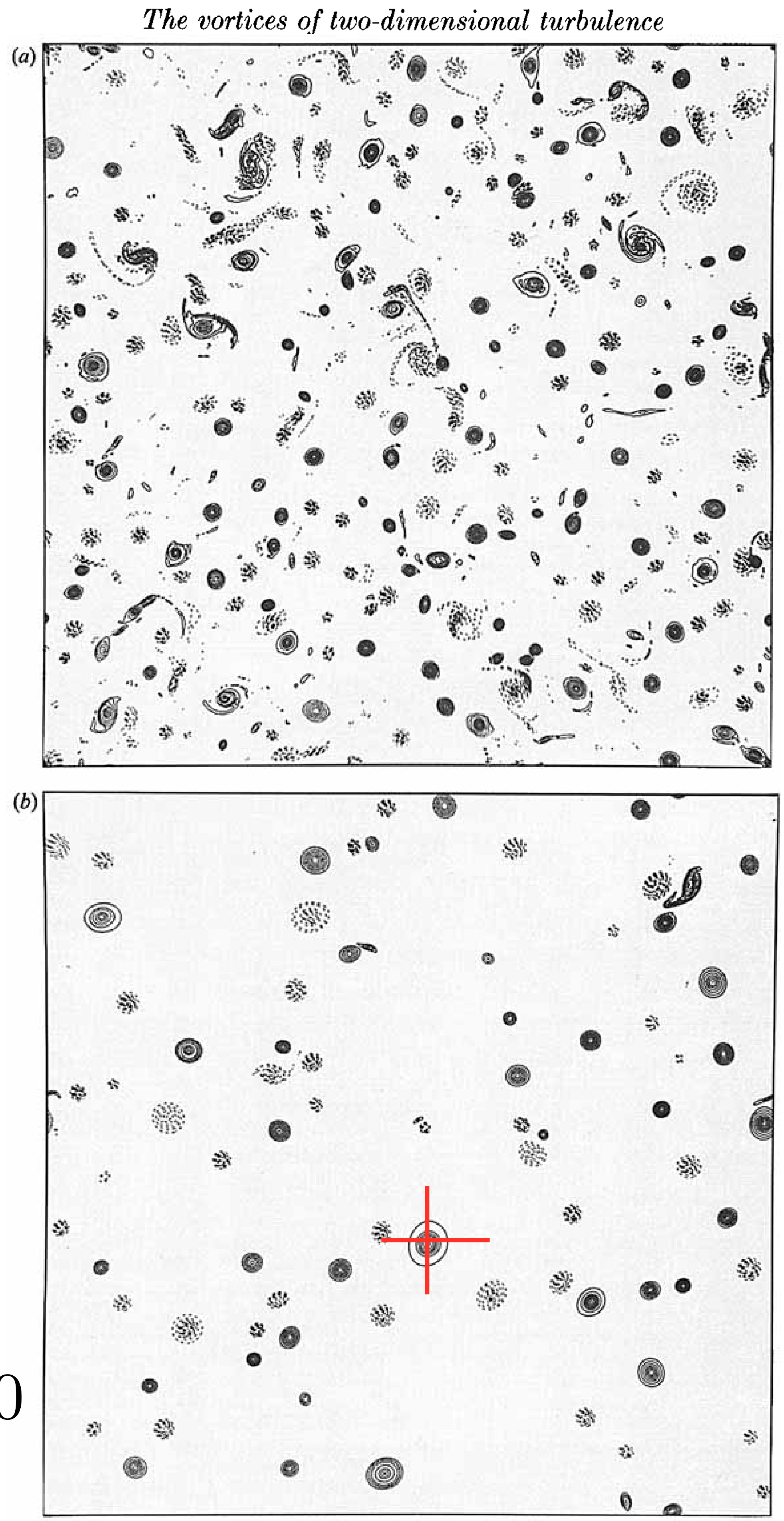
☛Like signed vortices merge into fewer and larger vortices.

☛Mergers jettison filaments into the chaotic sea of small-scale low-level vorticity.

☛Energy is conserved throughout the evolution.

McWilliams (JFM 1984 &1990)

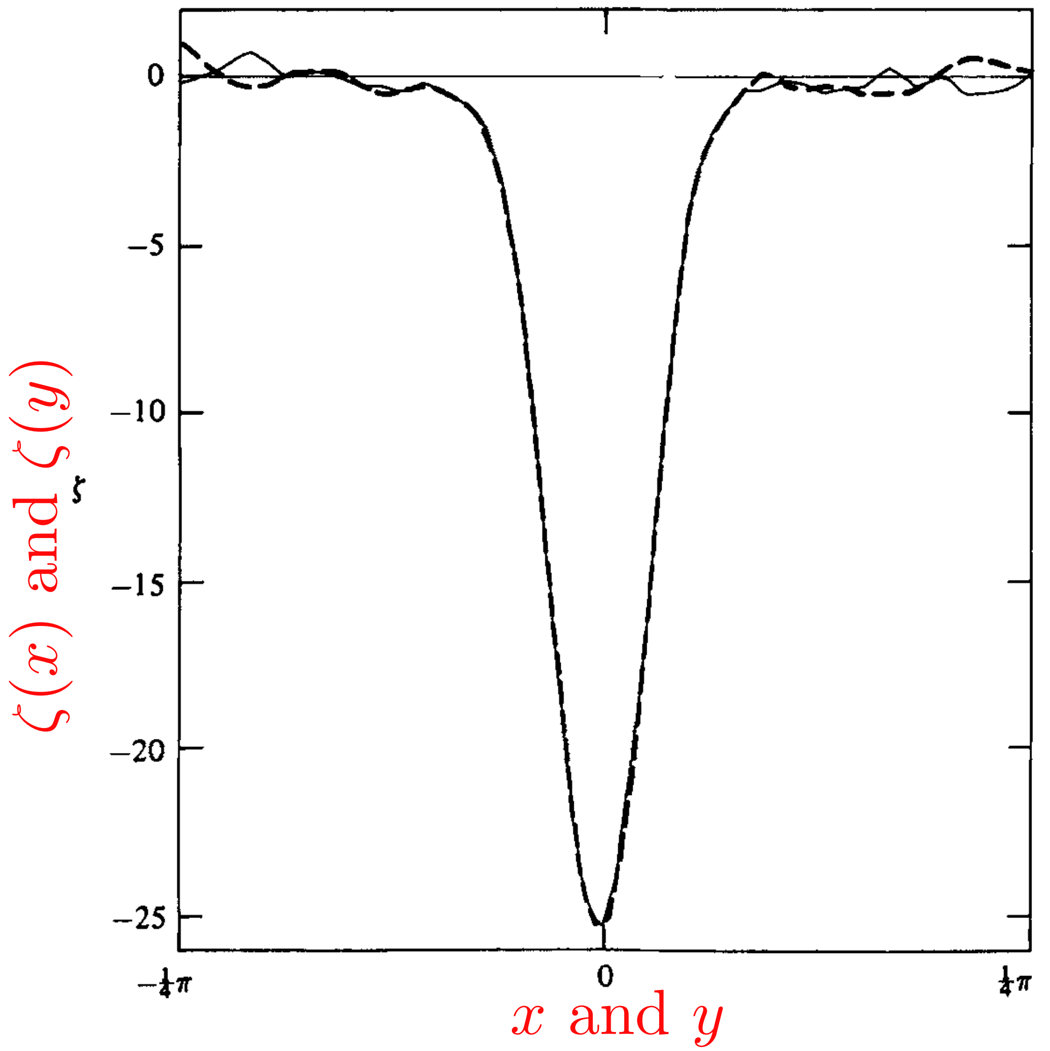
$t = 5$



$t = 20$

In isolation, the vortices are impressively axisymmetric.

They can be identified and counted by a vortex-census algorithm.



The initial emergence of vortices is not very well understood. But we can say something about the statistics of the emergent vortex gas.

The vortex census indicated that

$$\varrho(t) = \text{vortices per area} \sim t^{-\xi} \quad \text{with} \quad \xi \sim 0.71 - 0.75$$

Evolution of the vortex population

$\varrho(t) = \text{vortices per area} \sim t^{-\xi}$ with $\xi \sim 0.71 - 0.75$

☛ Follow Batchelor, and assume that energy is the only robustly conserved quantity, and use dimensional analysis.

$$\mathcal{E} = \frac{1}{(2\pi L)^2} \int \tfrac{1}{2} |\nabla \psi|^2 \, \mathrm{d}\boldsymbol{x}$$

$$\dim \mathcal{E} = \frac{L^2}{T^2}$$

The only dimensionally consistent expressions are:

$$\varrho \sim \frac{1}{\mathcal{E} t^2} \qquad \text{and} \qquad \mathcal{Z} \sim \frac{1}{t^2}$$

enstrophy

☛ Or try an analogy with colloidal aggregation:

$$\dot{\varrho}(t) = -\kappa \varrho^2 \qquad \Rightarrow \qquad \varrho(t) \sim \frac{1}{t}$$

The miserable failure of "energy scaling" means there must be other robustly conserved quantities.

$$\varrho(t) = \text{vortices per area} \sim t^{-\xi} \quad \text{with} \quad \xi \sim 0.71 - 0.75$$

versus

$$\varrho \sim \frac{1}{\mathcal{E} t^2}$$

Conservation of extrema in vortex cores

(Carnevale, McWilliams, Pomeau, Weiss & Young 1991)

There are two conserved quantities: ζ_{ext} and $\mathcal{E}$

Consequently there is a length and a time:

$$\ell \stackrel{\mathrm{def}}{=} \frac{\sqrt{\mathcal{E}}}{\zeta_{\mathrm{ext}}} \quad \text{and} \quad \tau \stackrel{\mathrm{def}}{=} \frac{1}{\zeta_{\mathrm{ext}}}$$

Dimensional analysis only tells us that:

$$\varrho(t) = \frac{1}{\ell^2} F\left(\frac{t}{\tau}\right)$$

What can we do?

Introduce: $a(t) = \text{typical vortex radius}$

Now consider:

$$\begin{aligned}\mathcal{E} &= \frac{1}{(2\pi L)^2}\int \tfrac{1}{2}|\nabla\psi|^2\,\mathrm{d}\boldsymbol{x} \\ &= -\frac{1}{(2\pi L)^2}\int \tfrac{1}{2}\psi\zeta\,\mathrm{d}\boldsymbol{x} \\ &= -\frac{1}{(2\pi L)^2}\int\!\mathrm{d}\boldsymbol{x}\int\!\mathrm{d}\boldsymbol{x}'\,\tfrac{1}{2}\zeta(\boldsymbol{x}')G(\boldsymbol{x}',\boldsymbol{x})\zeta(\boldsymbol{x}) \\ &\sim \varrho(t)\zeta_{\rm ext}^2 a(t)^4\end{aligned}$$

(Assume that all of the energy is due to the vortices.)

Energy is conserved:

$$\therefore\quad \varrho(t)\sim t^{-\xi}\qquad\Rightarrow\qquad a(t)\sim t^{\xi/4}$$

The circulation of a typical vortex is: $\Gamma(t)\sim\zeta_{\rm ext}a(t)^2\sim t^{\xi/2}$

Vorticity moments are:

$$\begin{aligned}\mathcal{Z}_n &= \frac{1}{(2\pi L)^2}\int \zeta^n\,\mathrm{d}\boldsymbol{x} \\ &\sim \varrho\,\zeta_{\rm ext}^n\,a^2 \\ &\sim t^{-\xi/2}\end{aligned}$$

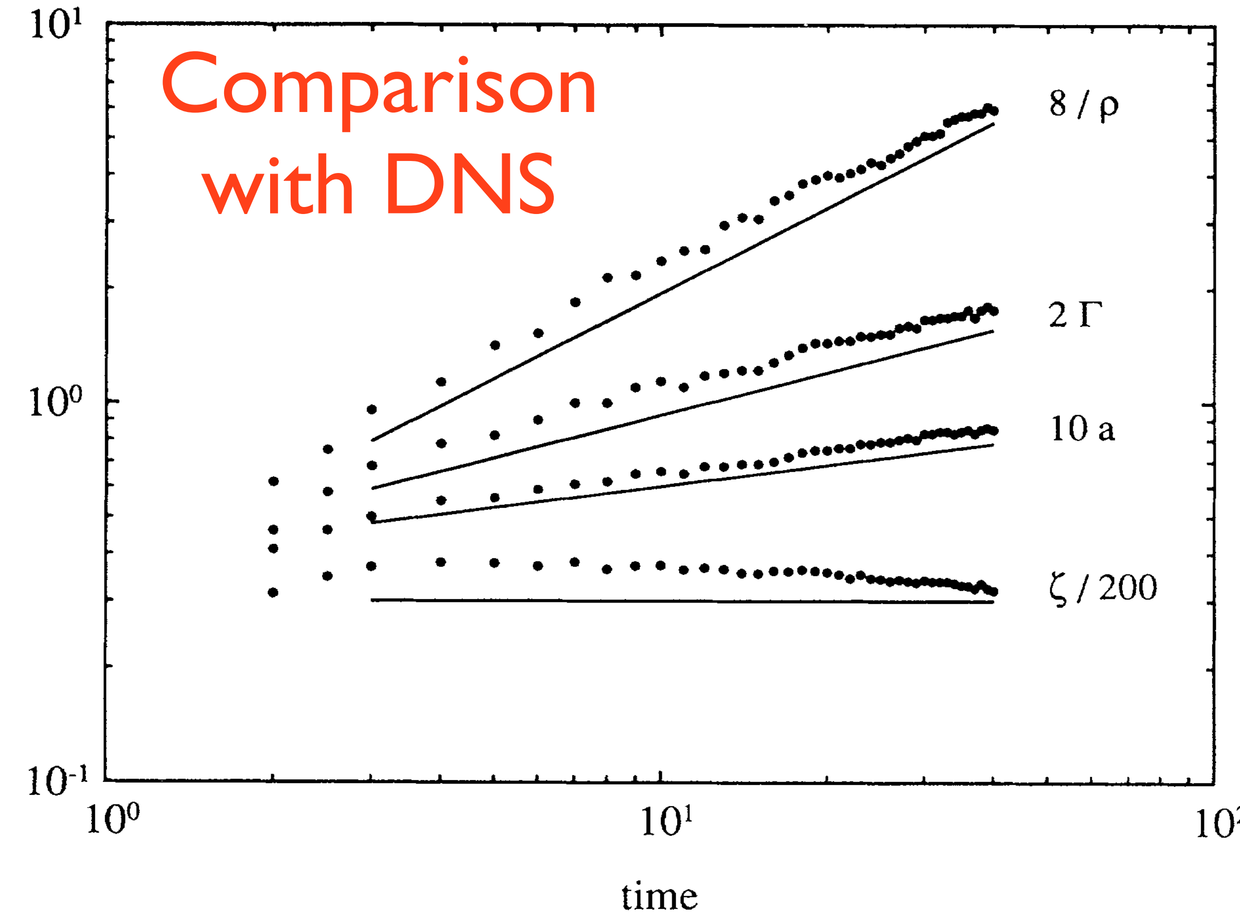
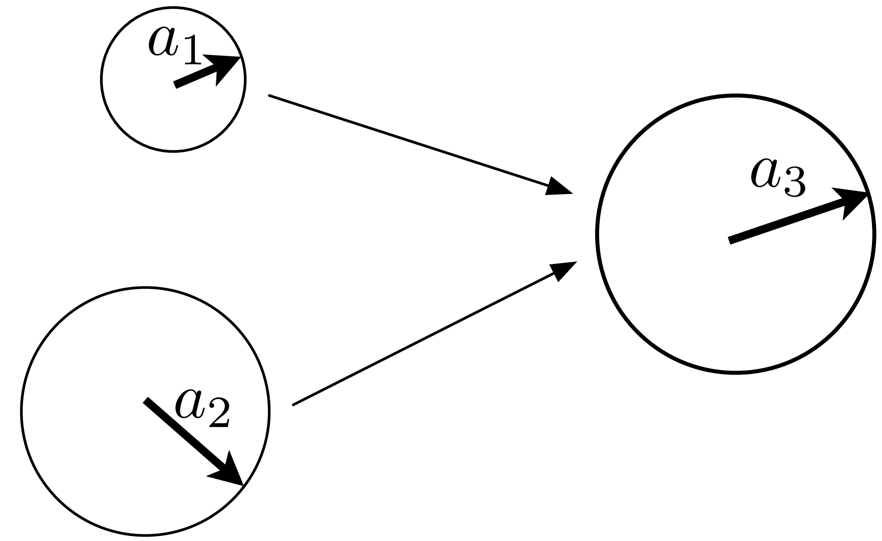


FIG. 1. Vortex data from a turbulence solution (Ref. 6): the reciprocal of the density of vortices ρ^{-1}, the mean absolute value of vortex circulation Γ, the mean vortex radius a, and the mean of the absolute value of the vorticity extrema ζ are shown. The solid straight-line segments show the predicted slopes based on the choice $\xi=0.75$, which is determined from the data for ρ^{-1}. The numerical factors for the data are chosen for display purposes only.

All statistical properties of the vortex gas can be expressed in terms of the exponent ξ.
These relations agree with DNS.
But there is no way to predict ξ.

As another test, we make a model based on “vortex patch” dynamics



A vortex patch moves like a point vortex, except when they get too close to another like-signed patch.

17 vortex trajectories in 2D turbulence

Trajectories of 17 point vortices

(Benzi, Paternello & Santangelo 1987)

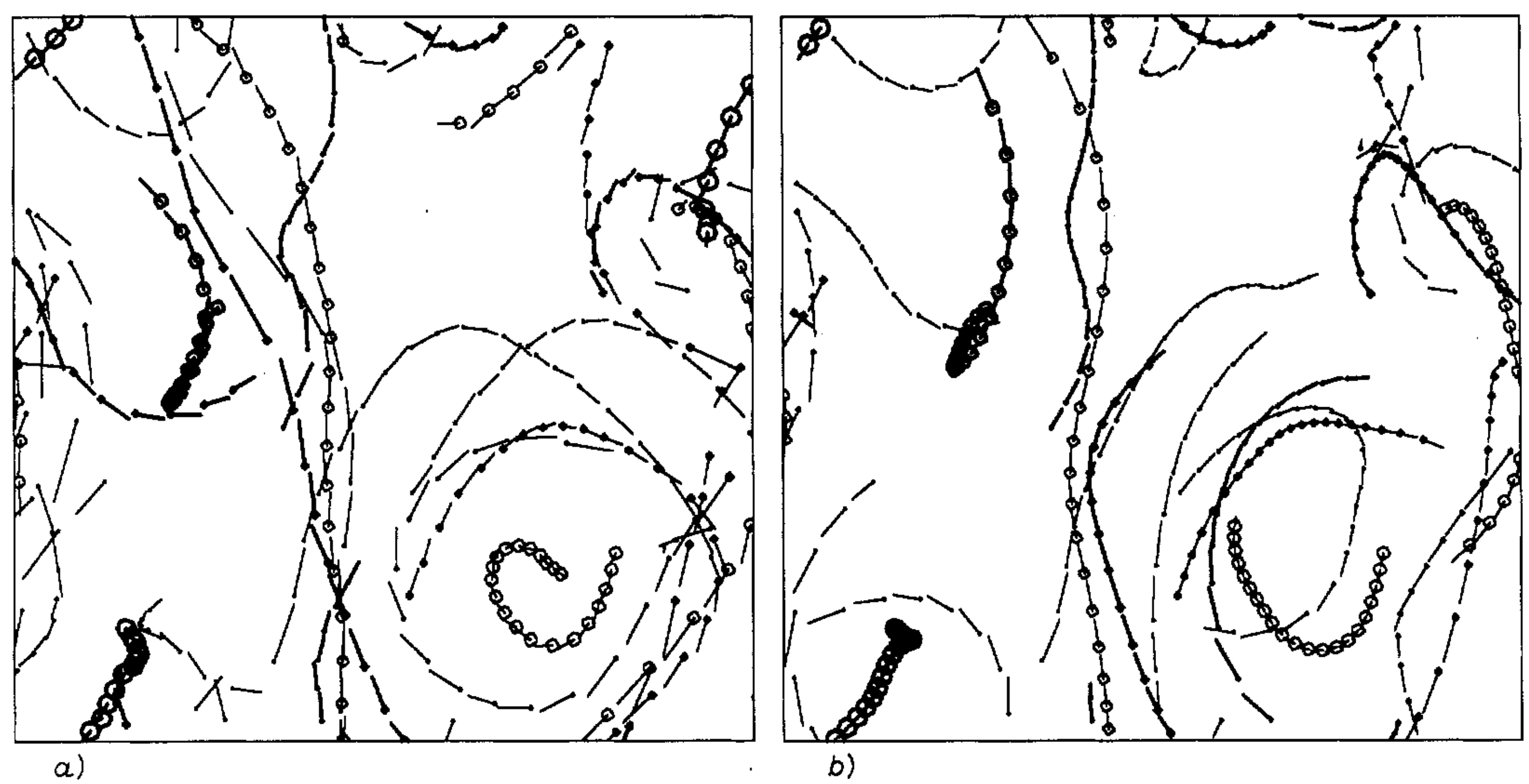
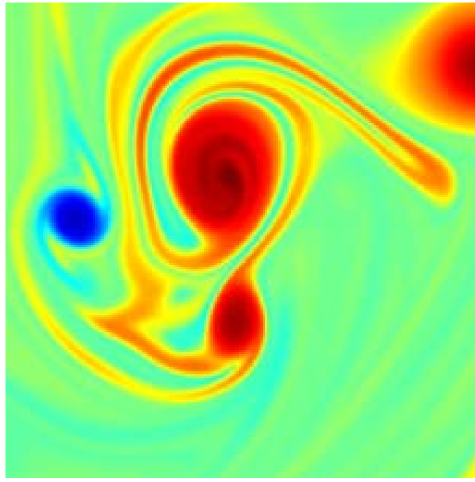


Fig. 3. – a) Trajectories of the centres of the 17 largest vortices (vortex area > 0.01); the plot has been obtained from the 512×512 simulation with the aid of a vortex recognition algorithm. The time interval plotted is $30 \div 40$. Thick (thin) lines are used for positive (negative) vortices. Circle sizes are proportional to vortex radii and segment lengths are proportional to velocities. b) The same as in a) but for the 17 point vortices, each one having the same total vorticity Γ_i of the corresponding vortex of the high-resolution simulation. The plot has been obtained from the solution of eqs. (2) starting with the positions of the centres of the corresponding vortices of fig. 1. Note the striking correspondence of the trajectories of a) and b).

The vortex-patch merger rule



(Carnevale, McWilliams, Pomeau, Weiss & Young 1991)

Two circular vortex "patches" merge at the critical separation

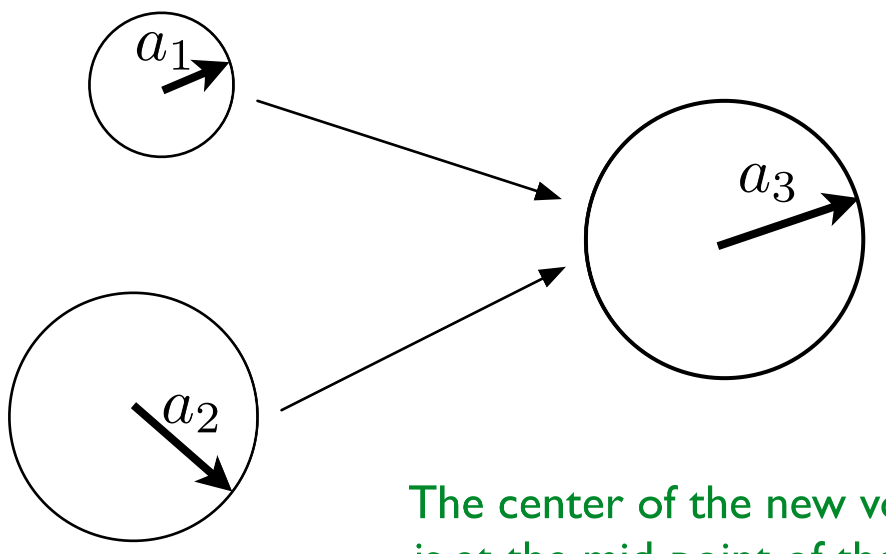
$$s_{\mathrm{crit}} = 1.65(a_1 + a_2)$$

The "merged vortex" has:

$$\zeta_{\mathrm{ext1}} = \zeta_{\mathrm{ext2}} = \zeta_{\mathrm{ext3}}$$

$$a_3^4 = a_1^4 + a_2^4$$

This rule encodes energy conservation (on average).



The center of the new vortex is at the mid-point of the line joining the original vortices.

Merger results in lost vortex area. This is an irreversible process occurring within the framework of the reversible system with $\nu=0$.

The vortex-patch model

Move circular top-hat vortex patches, as though they are point vortices concentrated at the patch center.

Close binary encounters result in merger into a bigger patch.

The crucial ingredient is that energy conservation is encoded into the merger rule.

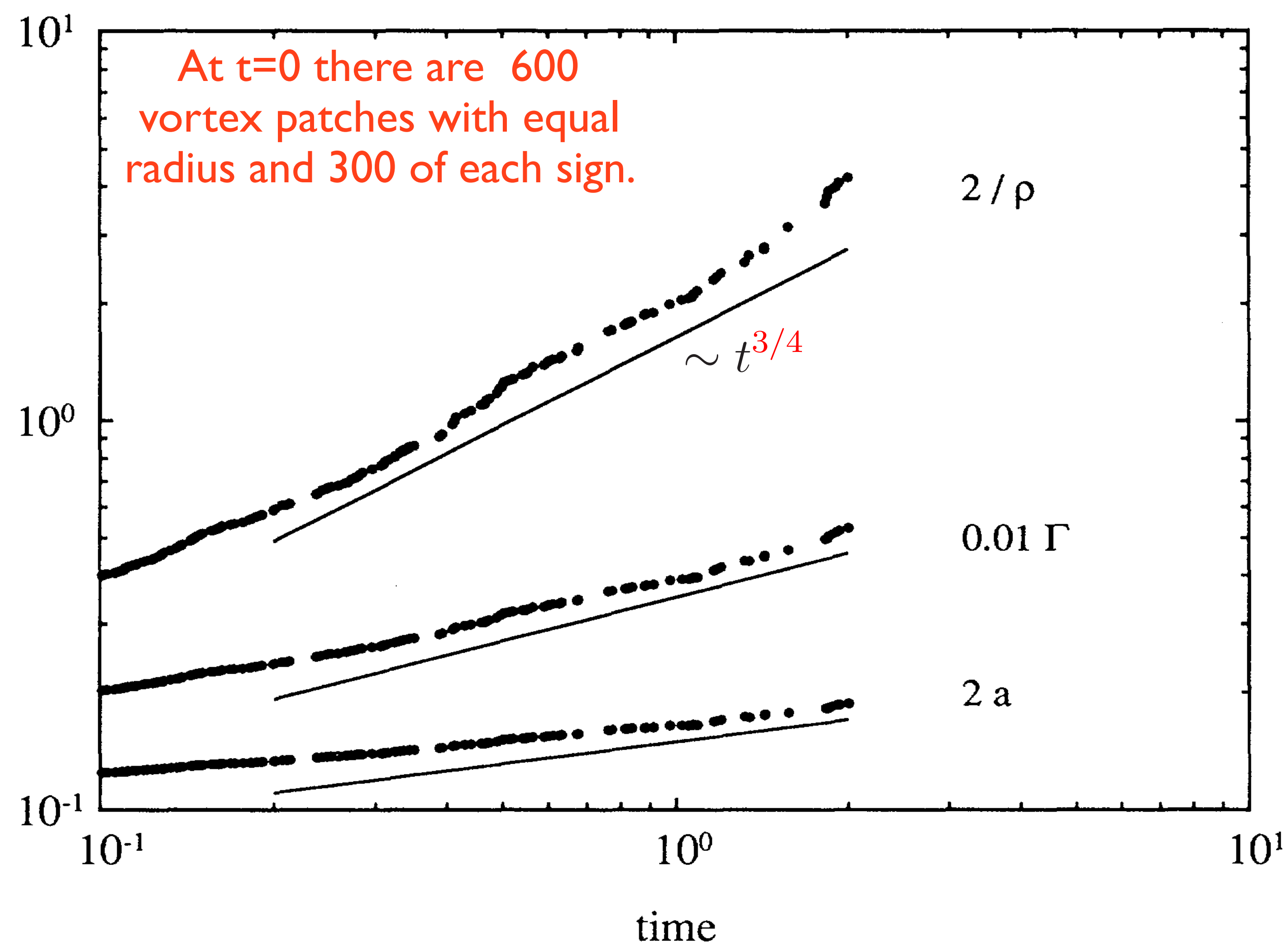
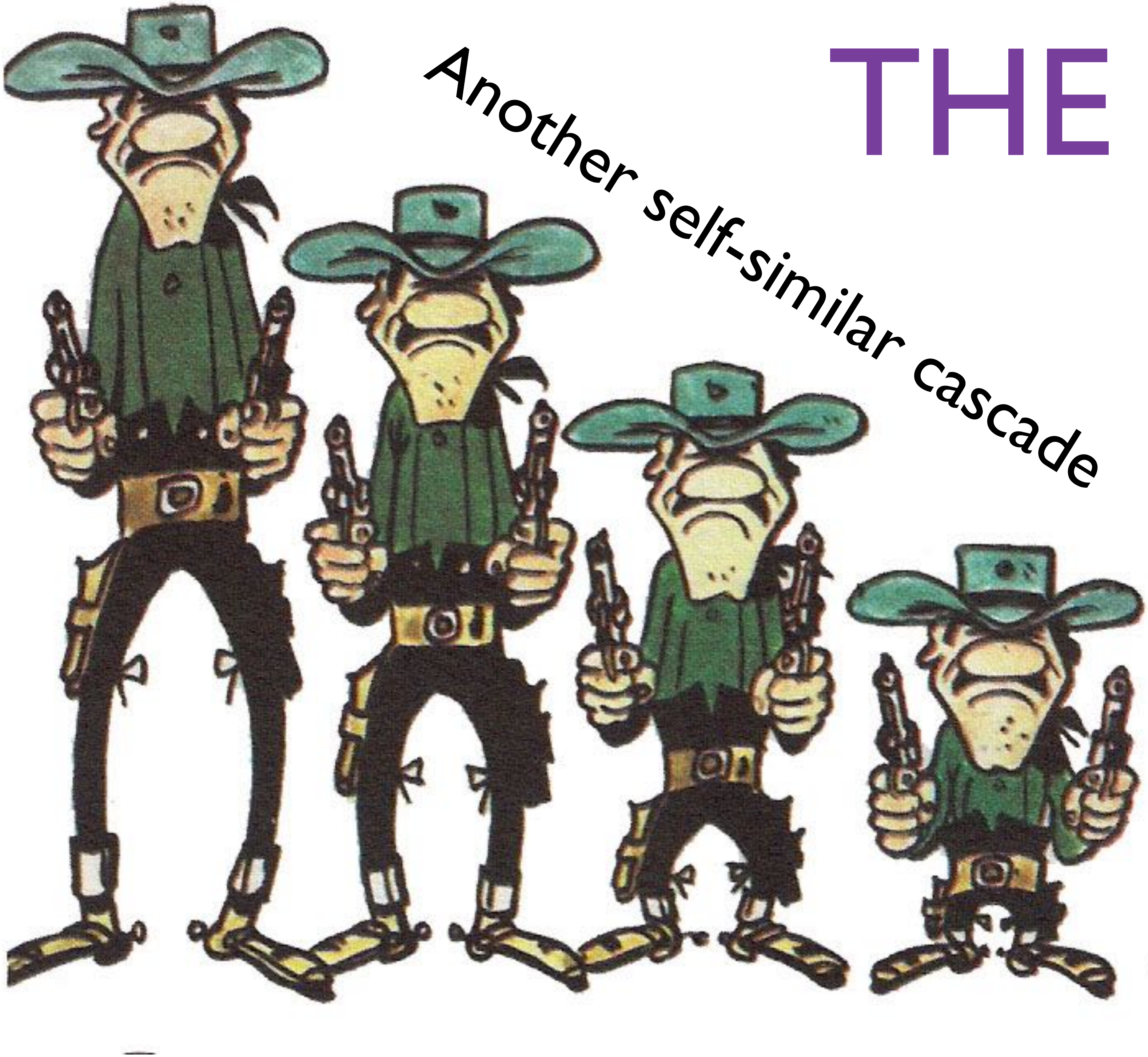


FIG. 2. Data from the modified point-vortex model. The format is as in Fig. 1. The value of mean vorticity extrema is not displayed here since it is constant in this model.

Vortex-patch statistics agree with the scaling theory, and the exponent is close to that of 2D turbulence.

THE END



Another self-similar cascade