Geometric generalised Lagrangian mean theories

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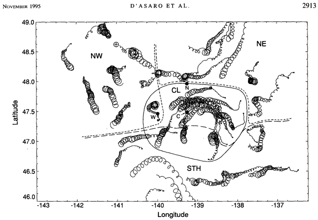
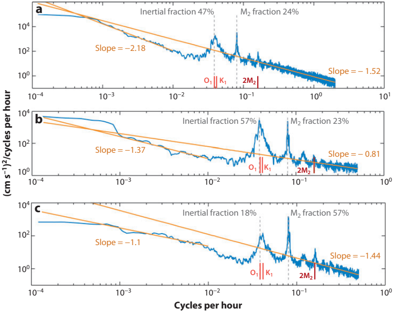
Wave–mean flow interactions

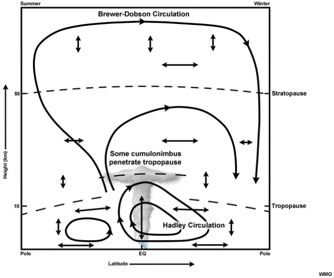
Separation between 'waves' and mean flows' in GFD:

- fast waves + slow motion,
- zonal mean + perturbation,
- resolved + unresolved.





Ferrari & Wunsch 2009



Zonal-mean atmospheric circulation

Wave–mean flow interactions

Main interest is for the evolution of the mean flow, but this is influenced by wave feedback.

Wave-mean flow theories have been developed to:

1. obtain simple governing equations for the mean,
2. include wave feedback terms that can parameterised,
3. track particle motion (e.g. for heat transport),
4. preserve geometric structures (vorticity/potential vorticity conservation, energy conservation, wave action),
5. be valid in multiple regimes (non-perturbative).

Important: for flows that are balanced (controlled by PV),

$$3 + 4 = 1$$

.

Lagrangian averaging.

Wave–mean flow interactions

Eulerian mean flow: does not track particle motion.

Example: zero-mean, time-periodic flow,

$$\boldsymbol{u} = \varepsilon \boldsymbol{U}(\boldsymbol{x}, t), \qquad \bar{\boldsymbol{u}}^{\mathrm{E}} = \langle \boldsymbol{U} \rangle = 0$$

Particle position: expanding $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1(t) + \varepsilon^2 \boldsymbol{x}_2(t) + \cdots$,

$$\begin{aligned} \varepsilon \dot{\boldsymbol{x}}_1 + \varepsilon^2 \dot{\boldsymbol{x}}_2 + \cdots &= \varepsilon \boldsymbol{U}(\boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1 + \cdots, t) \\ &= \varepsilon \boldsymbol{U}(\boldsymbol{x}_0, t) + \varepsilon^2 \boldsymbol{x}_1 \cdot \nabla \boldsymbol{U}(\boldsymbol{x}_0, t) + \cdots \end{aligned}$$

Order by order,

$$\boldsymbol{x}_1(t) = \boldsymbol{\xi}(t) = \int^t \boldsymbol{U}(\boldsymbol{x}_0, s)\, \mathrm{d}s : \text{periodic displacement},$$

$$\langle \dot{\boldsymbol{x}}_2(t) \rangle = \bar{\boldsymbol{u}}^{\mathrm{S}} = \langle \boldsymbol{\xi} \cdot \nabla \boldsymbol{U} \rangle : \text{Stokes drift}.$$

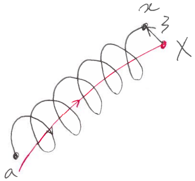
Wave–mean flow interactions

Generalised Lagrangian mean, GLM

Andrews & McIntyre 1978

Average 'following fluid particles':
fix particle label $\boldsymbol{a}$,

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{a}, t) + \boldsymbol{\xi}(\boldsymbol{X}(\boldsymbol{a}, t)) .$$



Define the mean flow by

$$\langle \boldsymbol{\xi} \rangle = 0 \quad \text{i.e.} \quad \boldsymbol{X}(\boldsymbol{a}, t) = \langle \boldsymbol{x}(\boldsymbol{a}, t) \rangle .$$

Lagrangian-mean velocity:

$$\dot{\boldsymbol{X}}(a, t) = \bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{X}, t) = \langle \boldsymbol{u}(\boldsymbol{X} + \boldsymbol{\xi}(\boldsymbol{X}, t), t) \rangle ,$$

Average equations of motion: see Bühler 2014

- nice mean vorticity equation,
- not-so-nice mean momentum equation.

Wave-mean flow interactions

Generalised Lagrangian mean

GLM is coordinate dependent: basic definitions make sense only in Euclidean space,

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{a}, t) + \boldsymbol{\xi}(\boldsymbol{X}(\boldsymbol{a}, t)), \quad \bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{X}, t) = \langle \boldsymbol{u}(\boldsymbol{X} + \boldsymbol{\xi}(\boldsymbol{X}, t), t)\rangle, \quad \langle \boldsymbol{\xi} \rangle = 0,$$

- cannot add points,
- cannot add vectors at different points on a manifold M (e.g. sphere),

This is damaging:

- $\boldsymbol{x} \in M$ but $\boldsymbol{X} \notin M$,
- $\nabla \cdot \boldsymbol{u} = 0$ but $\nabla \cdot \bar{\boldsymbol{u}}^{\mathrm{L}} \neq 0$.

Take a geometric approach:

- avoid temptation of coordinate dependence,
- results valid on arbitrary manifolds,
- GLM made easy(?).

Geometric approach

Notation

- use the flow map ϕ_t to avoid confusing maps and points,

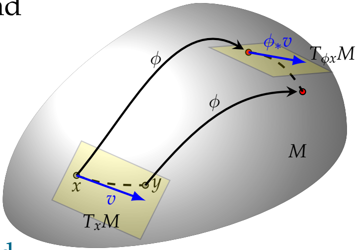
$$\boxed{x = \phi_t a}\,, \qquad \boxed{\dot{\phi}_t a = u(\phi_t a, t)}\,.$$

- use lowercases, $x \in M$, implicit time dependence $\phi = \phi_t$.

Main tools: push-forward, pull-back and Lie derivative

$$(\phi_* v)^i = v^j \partial_j \phi^i, \;\; \phi^* = (\phi^{-1})_*,$$

$$\mathcal{L}_u v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi_t)^* v.$$



Focus on incompressible perfect fluid: volume preserving, $\phi \in \mathrm{SDiff}(M)$.

Geometric approach

Notation

Consider an ensemble of flow maps $\phi = \phi^\alpha : M \to M$.

- $\alpha = 1, \cdots, N$,
- $\alpha \in [0, 2\pi]$, $\phi^\alpha(x, t, \varepsilon^{-1}t) = \Phi(x, t, \varepsilon^{-1}(t - \alpha))$,
- α, realisation of a flow-map-valued random process.

This defines an average for vectors and other linear objects:

$$\langle v^\alpha \rangle = N^{-1} \sum_{\alpha=1}^{N} v^\alpha, \quad \langle v^\alpha \rangle = \int v^\alpha \, \mathrm{d}\alpha.$$

Aim:

1. Define a a mean flow map: $\bar{\phi} \in \mathrm{SDiff}(M)$,
2. Derive dynamical equations for $\bar{\phi}$.

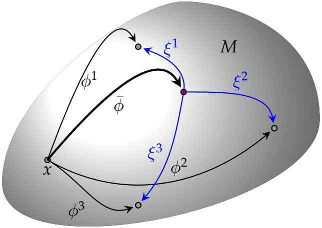
Start with 2.

Dynamics

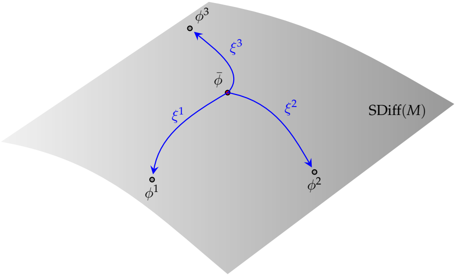
Decompose flow maps into mean and perturbation

$$\phi^\alpha = \xi^\alpha \circ \bar{\phi}.$$

with ξ^α an ensemble of perturbation maps. Holm 2000



Decomposition of the maps at one point x.



Decomposition of the maps in SDiff.

Dynamics

Good definition of $\bar{\phi}$:

- requires that ξ^α remain close to id for $t \gg 1$
- needs to be expressed in terms of ϕ^α or ξ^α, not u^α.

The mean velocity $\bar{u}$ is defined by

$$\dot{\bar{\phi}}x = \bar{u}(\bar{\phi}x), \quad \text{with } \bar{u} \neq \langle u^\alpha \rangle.$$

Chain rule:

$$\dot{\xi}^\alpha \circ (\xi^\alpha)^{-1} + \xi^\alpha_* \bar{u} = u^\alpha.$$

Deduce ξ^α when $\bar{\phi}$ and hence $\bar{u}$ are defined.

Dynamics

Write Euler equations in 'the right way':

$$\partial_t u + u \cdot \nabla u = -\nabla p \;\Leftrightarrow\; \partial_t u + u \cdot \nabla u + \nabla(u^2/2) = -\nabla(p - u^2/2).$$

Multiplying by $\mathrm{d}x$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u \cdot \mathrm{d}x) = -\mathrm{d}\pi.$$

Geometrically, define momentum:

- $\nu = u \cdot \mathrm{d}x$ in $\mathbb{R}^n$,
- $\boxed{\nu = g(u, \cdot) = u_\flat}$ on general M with metric $g(\cdot, \cdot)$.

Momentum is a one-form, dual to vector:

$$\nu(v) = \sum \nu_i v^i \in \mathbb{R}$$

($\nu = \nu_i \, \mathrm{d}x^i = g_{ij} u^j \, \mathrm{d}x^i$ covariant; $v = v^i \partial_{x^i}$ contravariant vector).

Euler equations:

$$\boxed{\partial_t \nu + \mathcal{L}_u \nu = -\mathrm{d}\pi}, \qquad \boxed{\operatorname{div} u = 0}.$$

Dynamics

$$\boxed{\partial_t \nu + \mathcal{L}_u \nu = -\mathrm{d}\pi}\,, \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^* \nu\right) = -\mathrm{d}\left(\phi^* \pi\right).$$

Why is this 'the right way'?

1. Kelvin's circulation theorem follows at once:

$$\oint_{\phi \mathcal{C}_0} \nu = \oint_{\mathcal{C}_0} \phi^* \nu = \text{const.}$$

2. The form emerges directly from the variational principle

$$\boxed{\min_{\phi \in \mathrm{SDiff}(M)} \int_0^T \mathrm{d}t \int_M g(u,u)\omega}\,.$$

   Euler equations: geodesic motion on $\mathrm{SDiff}(M)$. Arnold 1966

3. The alternative $\partial_t u + \nabla_u u = -\nabla p$ involves the covariant derivative ∇_u.

Dynamics

Mean dynamics: pull-back Euler equations with ξ^α, then average (on mean configuration $\bar{\phi}M$),

$$\langle \xi^{\alpha *} \left(\partial_t \nu^\alpha + \mathcal{L}_{u^\alpha} \nu^\alpha \right) \rangle = -\langle \xi^{\alpha *} \mathrm{d}\pi^\alpha \rangle \;\Leftrightarrow\; \partial_t \langle \xi^{\alpha *} \nu \rangle + \mathcal{L}_{\bar{u}} \langle \xi^{\alpha *} \nu \rangle = -\mathrm{d}(\cdots)$$

Define Lagrangian mean momentum: $\bar{\nu}^{\mathrm{L}} = \langle \xi^{\alpha *} \nu^\alpha \rangle$, then

$$\partial_t \bar{\nu}^{\mathrm{L}} + \mathcal{L}_{\bar{u}} \bar{\nu}^{\mathrm{L}} = -\mathrm{d}\bar{\pi}^{\mathrm{L}}.$$

Mean Kelvin theorem follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\bar{\phi}C_0} \bar{\nu}^{\mathrm{L}} = \text{const.}$$

Circulation of the Lagrangian-mean one-form $\bar{\nu}^{\mathrm{L}}$ along contours moving with velocity $\bar{u}$ is conserved

Dynamics

Mean flow

Wave-mean flow interaction = relation between $\bar{u}$ and $\bar{\nu}^{\mathrm{L}}$.

Pseudomomentum: $\boxed{-\mathsf{p} = \bar{\nu}^{\mathrm{L}} - g(\bar{u}, \cdot)}$.

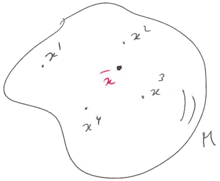
Closure: model to express p in terms of mean fields, $\bar{\nu}^{\mathrm{L}}$...
(e.g. linear waves, α-Euler).

Remarks:

- for more complex fluid models, $\overline{\cdot}^{\mathrm{L}} = \langle \xi^{\alpha *} \cdot \rangle$ is the natural averaging for: buoyancy, potential vorticity, magnetic field...,
- but $\bar{u} \neq \bar{u}^{\mathrm{L}}$.

Mean flow

Define $\bar{\phi}$: definition of an average on SDiff(M)



Natural to use:

- group structure,
- Riemannian structure.

Discuss 4 definitions:

1. extended GLM,
2. optimal transport,
3. geodesic,
4. Soward & Roberts' glm.

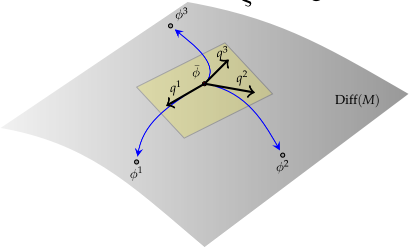
Mean flow

1. Extended GLM

$$\boxed{\bar{\phi} = \underset{\phi \in \mathrm{Diff}(M)}{\arg\min} \langle \int d^2(\phi, \phi^\alpha)\omega \rangle} \, .$$

Best defined in terms of s-dependent vector fields q^α such that

$$\xi^\alpha = \mathrm{e}^{\int_0^1 q_s^\alpha \, \mathrm{d}s} = \text{flow of } q_s^\alpha \text{ at } s = 1.$$



- $\partial_s q_s^\alpha + \nabla_{q_s^\alpha} q_s^\alpha = 0$,
- $\langle q_s^\alpha \rangle = 0$ at $s = 0$

defines the mean flow.

Perturbatively $q = q_1 + s q_2 + \cdots$ and $\xi^i(x) = x^i + \xi_1^i + \xi_2^i + \cdots$,

$$\boxed{\langle q_1 \rangle = 0}, \quad \boxed{\langle q_2 \rangle = -\nabla_{q_1} q_1}, \quad \langle \xi_1^i \rangle = 0, \quad \langle \xi_2^i \rangle = -\tfrac{1}{2}\Gamma^i_{jk} \langle \xi_1^j \xi_1^k \rangle.$$

Mean flow

2. Optimal transport

$$\bar{\phi} = \underset{\phi \in \mathrm{SDiff}(M)}{\arg\min} \, \langle \int d^2(\phi, \phi^\alpha)\omega \rangle .$$

As GLM, but with incompressibility constraint: $\bar{\phi}_* \omega = \omega$.

End condition: $\langle q_s^\alpha \rangle = \nabla \psi$ at $s = 0$ for some ψ. McCann 2001

Peturbatively:

$$\langle q_1 \rangle = 0, \quad \langle q_2 \rangle = -\mathsf{P} \langle \nabla_{q_1} q_1 \rangle,$$

$$\langle \xi_1^i \rangle = 0, \quad \langle \xi_2^i \rangle = \tfrac{1}{2}(\mathsf{I} - \mathsf{P}) \langle \xi_1^j \partial_j \xi_1^i \rangle - \tfrac{1}{2} \mathsf{P} \Gamma_{jk}^i \langle \xi_1^j \xi_1^k \rangle,$$

where P projection on divergence-free vector fields.

Mean flow

3. Geodesic

The Euler equations describe geodesics on $\mathrm{SDiff}(M)$ with metric

Arnold 1963

$$D^2(\phi,\psi) = \inf_{\gamma_s:[0,1]\to\mathrm{SDiff}(M)} \int_0^1 \int_M g(\dot\gamma_s,\dot\gamma_s)\omega \, \mathrm{d}s, \quad \gamma_0=\phi, \ \ \gamma_1=\psi.$$

Use this metric to define $\bar\phi$ as a Riemannian centre of mass:

$$\boxed{\bar\phi = \arg\min_{\phi\in\mathrm{SDiff}(M)} \langle D^2(\phi,\phi^\alpha)\rangle}\,.$$

- $\partial_s q_s^\alpha + \mathsf{P}\nabla_{q_s^\alpha} q_s^\alpha = 0$: Euler equations,
- $\langle q_s^\alpha\rangle = 0$ at $s=0$, end condition.

Pertubatively: $\boxed{\langle q_1\rangle = 0}$, $\boxed{\langle q_2\rangle = -\mathsf{P}\langle\nabla_{q_1} q_1\rangle}$, same as optimal transport to leading order.

Mean flow

Soward & Roberts 2010

4. glm

Take $q_s^\alpha = q^\alpha$ to be s-independent:

$$\xi^\alpha = \mathrm{e}^{q^\alpha} \quad \text{Lie group exponential,}$$

with

$$\langle q^\alpha \rangle = 0.$$

Perturbatively:

$$\langle q_1 \rangle = 0, \quad \langle q_2 \rangle = 0, \langle \xi_1^i \rangle = 0, \quad \langle \xi_2^i \rangle = \tfrac{1}{2} \langle \xi_1^j \partial_j \xi_1^i \rangle,$$

The simplest theory, but

- 'most' flows ξ^α cannot be written as exponentials,
- still usable perturbatively.

Application

Inertia-gravity-wave–mean flow interactions

Start with 3D rotating, Boussinesq equations,

$$\partial_t \nu_{\mathrm{a}}^{\alpha} + \mathcal{L}_{u^\alpha} \nu_{\mathrm{a}}^{\alpha} = -\mathrm{d}\pi^{\alpha} + \theta^{\alpha}\mathrm{d}z,$$
$$\partial_t \theta^{\alpha} + \mathcal{L}_{u^\alpha} \theta^{\alpha} = 0, \quad \operatorname{div} u^{\alpha} = 0,$$

with $\nu_{\mathrm{a}}^{\alpha} = \nu^{\alpha} + f(x\mathrm{d}y - y\mathrm{d}x)/2.$

PV (substance) conservation: Haynes & McIntyre 1990

$$\boxed{(\partial_t + \mathcal{L}_{u^\alpha})\, \mathrm{d}\nu_{\mathrm{a}}^{\alpha} \wedge \mathrm{d}\theta^{\alpha} = 0}$$

Lagrangian average: $\boxed{(\partial_t + \mathcal{L}_{\bar{u}})\, \mathrm{d}\overline{\nu_{\mathrm{a}}}^{\mathrm{L}} \wedge \mathrm{d}\overline{\theta}^{\mathrm{L}} = 0}$.

Application

Wave feedback of inertia-gravity waves

- assume $u^\alpha = \underbrace{u_1^\alpha}_{\text{fast waves}} + \varepsilon u_2^\alpha + \cdots$,
- take $\langle\cdot\rangle$ as fast-time average,
- $\bar{u}$ is geostrophically balanced: $\bar{u} = (-\bar{\psi}_y, \bar{\psi}_x, 0)$,
- mean momentum: $\bar{\nu}^{\mathrm{L}} = -\bar{\psi}_y \, \mathrm{d}x + \bar{\psi}_x \, \mathrm{d}y + \text{wave terms}$,
- mean dynamics is controlled by Lagrangian-mean PV:

$$\partial_t \bar{q}^{\mathrm{L}} + \partial(\bar{\psi}, \bar{q}^{\mathrm{L}}) = 0,$$

$$\bar{q}^{\mathrm{L}} = \left(\nabla^2 + \frac{f^2}{N^2}\right)\bar{\psi} + \langle \partial(u_1, \xi_1) + \partial(v_1, \eta_1)\rangle + f\langle \partial(\xi_1, \eta_1)\rangle + f\nabla \cdot \langle \boldsymbol{\xi}_1 \cdot \nabla \boldsymbol{\xi}_1\rangle/2.$$

Holmes–Cerfon et al 2011, Xie & V 2015, Wagner & Young 2015, Salmon 2016

Conclusion

- Revisit Andrews & McIntyre's GLM using geometric formulation to
  - obtain an incompressible mean flow,
  - mean trajectories constrained to M,
  - coordinate independence.
- natural definition of Lagrangian mean in terms of pull-back: $\bar{\tau}^{\mathrm{L}} = \langle \xi^* \tau \rangle$,
- several definitions of the mean flow, $O(\varepsilon^2)$ apart,
- mean circulation theorem is automatic,
- relation between $\bar{u}$ and $\bar{\nu}^{\mathrm{L}}$ encodes wave-mean flow interactions,
- geodesic GLM + Taylor closure: Holm's α-model. Oliver 2017