Geometric generalised Lagrangian mean theories

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Wave–mean flow interactions

Separation between ‘waves’ and mean flows’ in GFD:

- fast waves + slow motion,
- zonal mean + perturbation,
- resolved + unresolved.

Ferrari & Wunsch 2009
Wave–mean flow interactions

Main interest is for the evolution of the mean flow, but this is influenced by wave feedback.

Wave-mean flow theories have been developed to:

1. obtain simple governing equations for the mean,
2. include wave feedback terms that can parameterised,
3. track particle motion (e.g. for heat transport),
4. preserve geometric structures (vorticity/potential vorticity conservation, energy conservation, wave action),
5. be valid in multiple regimes (non-perturbative).

Important: for flows that are balanced (controlled by PV),

\[ 3 + 4 = 1 \]

Lagrangian averaging.
Wave–mean flow interactions

Eulerian mean flow: does not track particle motion.

Example: zero-mean, time-periodic flow,

\[ u = \varepsilon U(x, t), \quad \bar{u}^E = \langle U \rangle = 0 \]

Particle position: expanding \( x(t) = x_0 + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \),

\[ \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots = \varepsilon U(x_0 + \varepsilon x_1 + \cdots, t) \]
\[ = \varepsilon U(x_0, t) + \varepsilon^2 x_1 \cdot \nabla U(x_0, t) + \cdots \]

Order by order,

\[ x_1(t) = \xi(t) = \int_0^t U(x_0, s) \, ds : \text{periodic displacement,} \]

\[ \langle \dot{x}_2(t) \rangle = \bar{u}^S = \langle \xi \cdot \nabla U \rangle : \text{Stokes drift.} \]
Wave–mean flow interactions

Generalised Lagrangian mean, GLM

Average ‘following fluid particles’: fix particle label $a$,

$$x = X(a, t) + \xi(X(a, t)) .$$

Define the mean flow by

$$\langle \xi \rangle = 0 \quad \text{i.e.} \quad X(a, t) = \langle x(a, t) \rangle .$$

Lagrangian-mean velocity:

$$\dot{X}(a, t) = \bar{u}^L(X, t) = \langle u(X + \xi(X, t), t) \rangle ,$$

Average equations of motion:

- nice mean vorticity equation,
- not-so-nice mean momentum equation.
Wave-mean flow interactions

Generalised Lagrangian mean

GLM is coordinate dependent: basic definitions make sense only in Euclidean space,

\[ x = X(a, t) + \xi(X(a, t)) \quad \text{and} \quad \bar{u}^L(X, t) = \langle u(X + \xi(X, t), t) \rangle, \quad \langle \xi \rangle = 0, \]

- cannot add points,
- cannot add vectors at different points on a manifold \( M \) (e.g. sphere),

This is damaging:

- \( x \in M \) but \( X \notin M \),
- \( \nabla \cdot u = 0 \) but \( \nabla \cdot \bar{u}^L \neq 0 \).

Take a geometric approach:

- avoid temptation of coordinate dependence,
- results valid on arbitrary manifolds,
- GLM made easy(?).
Geometric approach

Notation

- use the flow map \( \phi_t \) to avoid confusing maps and points,

\[
x = \phi_t a, \quad \dot{\phi}_t a = u(\phi_t a, t).
\]

- use lowercases, \( x \in M \), implicit time dependence \( \phi = \phi_t \).

Main tools: push-forward, pull-back and Lie derivative

\[
(\phi_*v)^i = v^j \partial_j \phi^i, \quad \phi^* = (\phi^{-1})_*,
\]

\[
\mathcal{L}_u v = \left. \frac{d}{dt} \right|_{t=0} (\phi_t)^* v.
\]

Focus on incompressible perfect fluid: volume preserving, \( \phi \in \text{SDiff}(M) \).
Geometric approach

Notation

Consider an ensemble of flow maps $\phi = \phi^\alpha : M \to M$.

- $\alpha = 1, \ldots, N$,
- $\alpha \in [0, 2\pi]$, $\phi^\alpha(x, t, \varepsilon^{-1}t) = \Phi(x, t, \varepsilon^{-1}(t - \alpha))$,
- $\alpha$, realisation of a flow-map-valued random process.

This defines an average for vectors and other linear objects:

$$\langle v^\alpha \rangle = N^{-1} \sum_{\alpha=1}^{N} v^\alpha, \quad \langle v^\alpha \rangle = \int v^\alpha \, d\alpha.$$

Aim:

1. Define a mean flow map: $\bar{\phi} \in \text{SDiff}(M)$,
2. Derive dynamical equations for $\bar{\phi}$.

Start with 2.
Dynamics

Decompose flow maps into mean and perturbation

\[ \phi^\alpha = \xi^\alpha \circ \bar{\phi} \]

with \( \xi^\alpha \) an ensemble of perturbation maps. Holm 2000

Decomposition of the maps at one point \( x \).

Decomposition of the maps in \( \text{SDiff} \).
Dynamics

Good definition of $\bar{\phi}$:
- requires that $\xi^\alpha$ remain close to id for $t \gg 1$
- needs to be expressed in terms of $\phi^\alpha$ or $\xi^\alpha$, not $u^\alpha$.

The mean velocity $\bar{u}$ is defined by

$$\dot{\bar{\phi}} x = \bar{u}(\bar{\phi} x) , \text{ with } \bar{u} \neq \langle u^\alpha \rangle.$$

Chain rule:

$$\dot{\xi}^\alpha \circ (\xi^\alpha)^{-1} + \xi_*^\alpha \bar{u} = u^\alpha.$$

Deduce $\xi^\alpha$ when $\bar{\phi}$ and hence $\bar{u}$ are defined.
Dynamics

Write Euler equations in ‘the right way’:
\[ \partial_t u + u \cdot \nabla u = -\nabla p \Leftrightarrow \partial_t u + u \cdot \nabla u + \nabla (u^2/2) = -\nabla (p - u^2/2). \]

Multiplying by \( dx \):
\[ \frac{d}{dt} (u \cdot dx) = -d\pi. \]

Geometrically, define momentum:
\[ \nu = u \cdot dx \text{ in } \mathbb{R}^n, \]
\[ \nu = g(u, \cdot) = u_b \text{ on general } M \text{ with metric } g(\cdot, \cdot). \]

Momentum is a one-form, dual to vector:
\[ \nu (v) = \sum \nu_i v^i \in \mathbb{R} \]
\[ (\nu = \nu_i dx^i = g_{ij} u^j dx^i \text{ covariant}; \nu = v^i \partial_x^i \text{ contravariant vector}). \]

Euler equations:
\[ \partial_t \nu + \mathcal{L}_u \nu = -d\pi, \quad \text{div } u = 0. \]
Dynamics

\[ \partial_t \nu + \mathcal{L}_u \nu = -d \pi , \quad \text{i.e.,} \quad \frac{d}{dt} (\phi^* \nu) = -d (\phi^* \pi) . \]

Why is this ‘the right way’?

1. Kelvin’s circulation theorem follows at once:

\[ \oint_{\phi C_0} \nu = \oint_{C_0} \phi^* \nu = \text{const} . \]

2. The form emerges directly from the variational principle

\[ \min_{\phi \in \text{SDiff}(M)} \int_0^T dt \int_M g(u, u) \omega . \]

Euler equations: geodesic motion on \( \text{SDiff}(M) \). Arnold 1966

3. The alternative \( \partial_t u + \nabla_u u = -\nabla p \) involves the covariant derivative \( \nabla_u \).
Dynamics

Mean dynamics: pull-back Euler equations with $\xi^\alpha$, then average (on mean configuration $\bar{\phi}M$),

$$\langle \xi^{\alpha*} (\partial_t \nu^\alpha + L_{\bar{u}^\alpha} \nu^\alpha) \rangle = -\langle \xi^{\alpha*} d\pi^\alpha \rangle \iff \partial_t \langle \xi^{\alpha*} \nu \rangle + L_{\bar{u}} \langle \xi^{\alpha*} \nu \rangle = -d(\cdots)$$

Define Lagrangian mean momentum: $\bar{\nu}^L = \langle \xi^{\alpha*} \nu^\alpha \rangle$, then

$$\partial_t \bar{\nu}^L + L_{\bar{u}} \bar{\nu}^L = -d\bar{\pi}^L.$$

Mean Kelvin theorem follows:

$$\frac{d}{dt} \int_{\bar{\phi}C_0} \bar{\nu}^L = \text{const.}$$

Circulation of the Lagrangian-mean one-form $\bar{\nu}^L$ along contours moving with velocity $\bar{u}$ is conserved
Dynamics

Mean flow

Wave-mean flow interaction = relation between $\bar{u}$ and $\bar{\nu}^L$.

Pseudomomentum: $-p = \bar{\nu}^L - g(\bar{u}, \cdot)$.

Closure: model to express $p$ in terms of mean fields, $\bar{\nu}^L$… (e.g. linear waves, $\alpha$-Euler).

Remarks:

▶ for more complex fluid models, $\bar{\cdot}^L = \langle \xi^{\alpha} \cdot \rangle$ is the natural averaging for: buoyancy, potential vorticity, magnetic field…,

▶ but $\bar{u} \neq \bar{u}^L$. 
Mean flow

Define $\bar{\phi}$: definition of an average on $\text{SDiff}(M)$

Natural to use:
- group structure,
- Riemannian structure.

Discuss 4 definitions:
1. extended GLM,
2. optimal transport,
3. geodesic,
4. Soward & Roberts' glm.
Mean flow

1. Extended GLM

\[ \bar{\phi} = \arg \min_{\phi \in \text{Diff}(M)} \int d^2(\phi, \phi^\alpha) \omega. \]

Best defined in terms of \( s \)-dependent vector fields \( q^\alpha \) such that

\[ \xi^\alpha = e^{\int_0^1 q_s^\alpha ds} = \text{flow of } q_s^\alpha \text{ at } s = 1. \]

- \( \partial_s q_s^\alpha + \nabla q_s^\alpha q_s^\alpha = 0, \)
- \( \langle q_s^\alpha \rangle = 0 \text{ at } s = 0 \)
  defines the mean flow.

Perturbatively \( q = q_1 + sq_2 + \cdots \) and \( \xi^i(x) = x^i + \xi_1^i + \xi_2^i + \cdots, \)

\[ \langle q_1 \rangle = 0, \quad \langle q_2 \rangle = -\nabla q_1 q_1, \quad \langle \xi_1^i \rangle = 0, \quad \langle \xi_2^i \rangle = -\frac{1}{2} \Gamma_{jk}^i \langle \xi_1^j \xi_1^k \rangle. \]
Mean flow

2. Optimal transport

\[ \bar{\phi} = \arg \min_{\phi \in \text{SDiff}(M)} \langle \int d^2(\phi, \phi^\alpha) \omega \rangle. \]

As GLM, but with incompressibility constraint: \( \bar{\phi}_* \omega = \omega \).

End condition: \( \langle q^\alpha_s \rangle = \nabla \psi \) at \( s = 0 \) for some \( \psi \). McCann 2001

Perturbatively:

\[ \langle q_1 \rangle = 0, \quad \langle q_2 \rangle = -P \langle \nabla q_1 q_1 \rangle, \]

\[ \langle \xi^i_1 \rangle = 0, \quad \langle \xi^i_2 \rangle = \frac{1}{2} (I - P) \langle \xi^j_1 \partial_j \xi^i_1 \rangle - \frac{1}{2} P \Gamma^i_{jk} \langle \xi^j_1 \xi^k_1 \rangle, \]

where \( P \) projection on divergence-free vector fields.
Mean flow

3. Geodesic

The Euler equations describe geodesics on SDiff(M) with metric

\[ D^2(\phi, \psi) = \inf_{\gamma_s: [0,1] \to \text{SDiff}(M)} \int_0^1 \int_M g(\dot{\gamma}_s, \dot{\gamma}_s) \omega \, ds, \quad \gamma_0 = \phi, \quad \gamma_1 = \psi. \]

Use this metric to define \( \bar{\phi} \) as a Riemannian centre of mass:

\[ \bar{\phi} = \arg \min_{\phi \in \text{SDiff}(M)} \langle D^2(\phi, \phi^\alpha) \rangle. \]

- \( \partial_s q_s^\alpha + P \nabla q_s^\alpha q_s^\alpha = 0 \): Euler equations,
- \( \langle q_s^\alpha \rangle = 0 \) at \( s = 0 \), end condition.

Pertubatively: \( \langle q_1 \rangle = 0 \), \( \langle q_2 \rangle = -P \langle \nabla q_1 q_1 \rangle \), same as optimal transport to leading order.
Mean flow

Take $q_s^\alpha = q^\alpha$ to be $s$-independent:

$$\xi^\alpha = e^{q^\alpha} \quad \text{Lie group exponential,}$$

with

$$\langle q^\alpha \rangle = 0.$$

Perturbatively:

$$\langle q_1 \rangle = 0, \quad \langle q_2 \rangle = 0, \langle \xi^i_1 \rangle = 0, \quad \langle \xi^i_2 \rangle = \frac{1}{2} \langle \xi^j_1 \partial_j \xi^i_1 \rangle,$$

The simplest theory, but

- ‘most’ flows $\xi^\alpha$ cannot be written as exponentials,
- still usable perturbatively.
Application
Inertia-gravity-wave–mean flow interactions

Start with 3D rotating, Boussinesq equations,

\[
\begin{align*}
\partial_t \nu^\alpha_a + \mathcal{L}u^\alpha \nu^\alpha_a &= -d\pi^\alpha + \theta^\alpha \, dz, \\
\partial_t \theta^\alpha + \mathcal{L}u^\alpha \theta^\alpha &= 0, \\
\text{div } u^\alpha &= 0,
\end{align*}
\]

with \( \nu^\alpha_a = \nu^\alpha + f(xdy - ydx)/2. \)

PV (substance) conservation: Haynes & McIntyre 1990

\[
(\partial_t + \mathcal{L}u^\alpha) \, d\nu^\alpha_a \wedge d\theta^\alpha = 0
\]

Lagrangian average:

\[
(\partial_t + \mathcal{L}\bar{u}) \, d\nu^L_a \wedge d\bar{\theta}^L = 0.
\]
Application

Wave feedback of inertia-gravity waves

- assume \( u^\alpha = u_1^\alpha + \varepsilon u_2^\alpha + \cdots \),
- fast waves
- take \( \langle \cdot \rangle \) as fast-time average,
- \( \bar{u} \) is geostrophically balanced: \( \bar{u} = (-\bar{\psi}_y, \bar{\psi}_x, 0) \),
- mean momentum: \( \bar{\nu}^L = -\bar{\psi}_y \, dx + \bar{\psi}_x \, dy + \) wave terms,
- mean dynamics is controlled by Lagrangian-mean PV:
  \[
  \partial_t \bar{q}^L + \partial (\bar{\psi}, \bar{q}^L) = 0,
  \]
  \[
  \bar{q}^L = \left( \nabla^2 + \frac{f^2}{N^2} \right) \bar{\psi} 
  + \langle \partial(u_1, \xi_1) + \partial(v_1, \eta_1) \rangle + f \langle \partial(\xi_1, \eta_1) \rangle + f \nabla \cdot \langle \xi_1 \cdot \nabla \xi_1 \rangle / 2.
  \]

Conclusion

- Revisit Andrews & McIntyre’s GLM using geometric formulation to
  - obtain an incompressible mean flow,
  - mean trajectories constrained to $M$,
  - coordinate independence.
- natural definition of Lagrangian mean in terms of pull-back: $\bar{\tau}^L = \langle \xi^* \tau \rangle$,
- several definitions of the mean flow, $O(\varepsilon^2)$ apart,
- mean circulation theorem is automatic,
- relation between $\bar{u}$ and $\bar{v}^L$ encodes wave-mean flow interactions,
- geodesic GLM + Taylor closure: Holm’s $\alpha$-model. Oliver 2017