A fluctuation-dissipation relation for the ocean subject to turbulent atmospheric forcing

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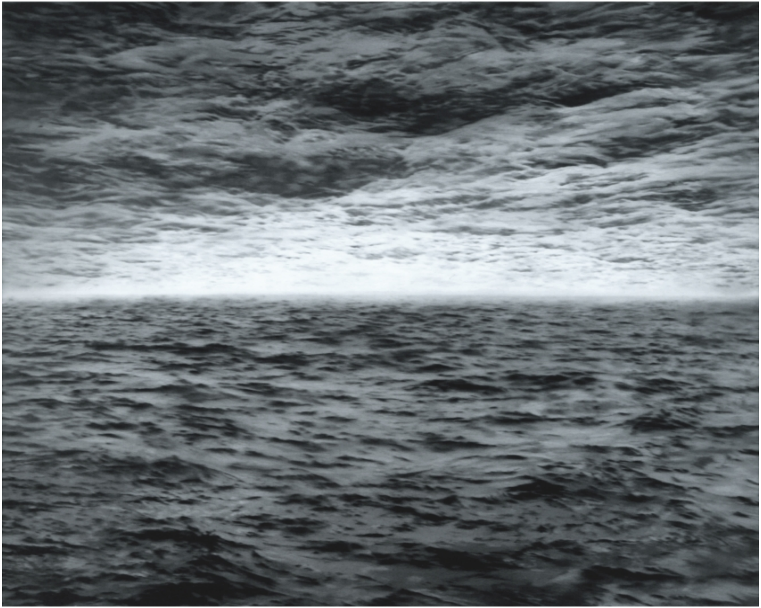
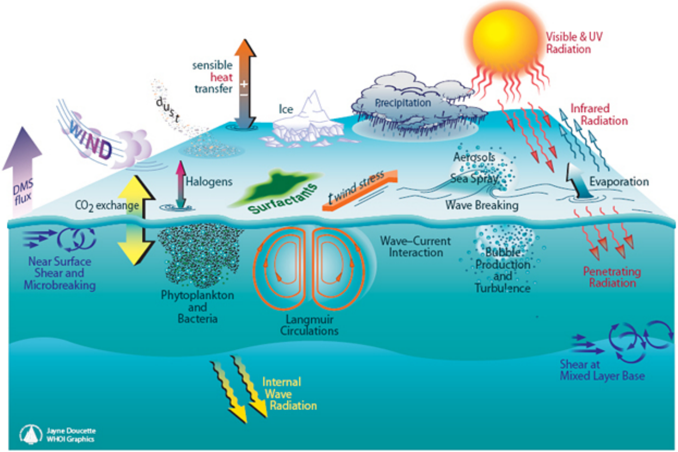
Air-Sea Interaction

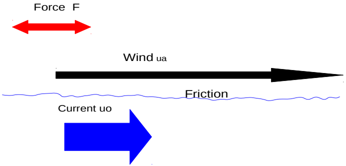


Seestück (G. Richter)

Air-Sea Interaction



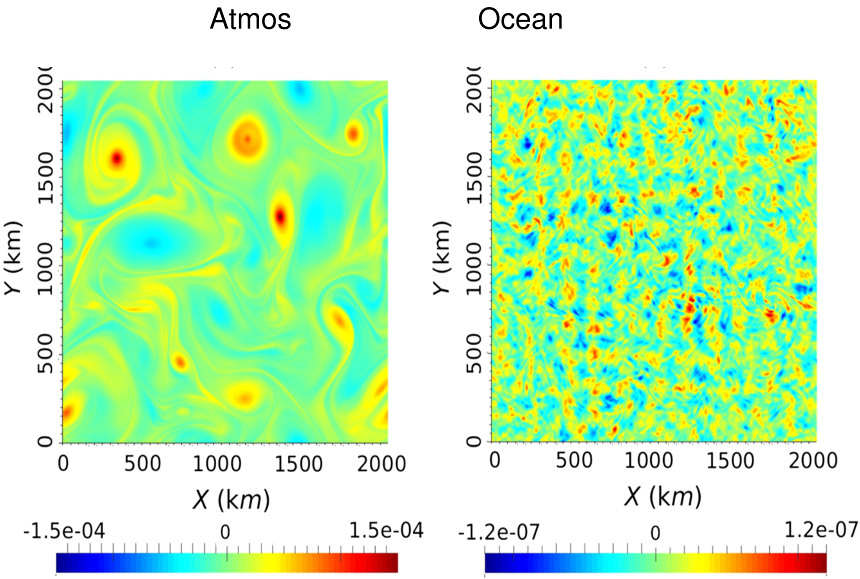
Model



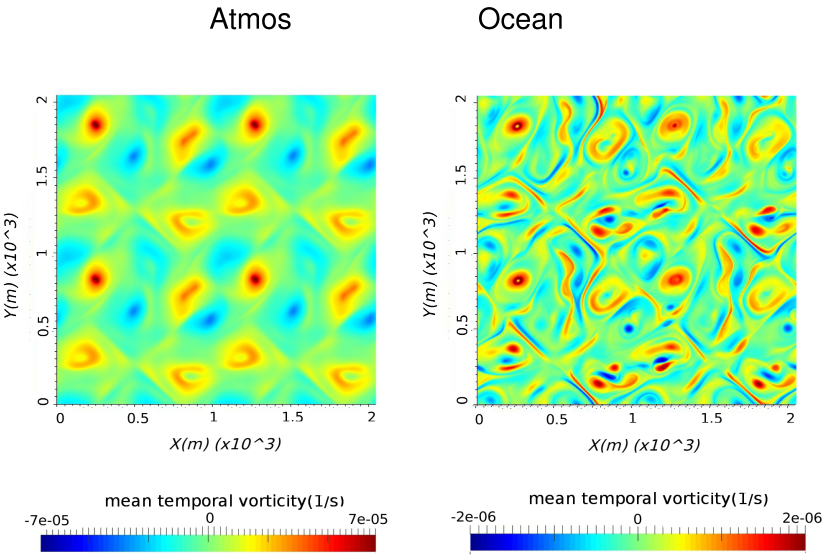
Parameters :

- mass ratio ocean/atmosphere: m
- friction coefficient (nonlinear): c_D

Variability



Variability



Variability

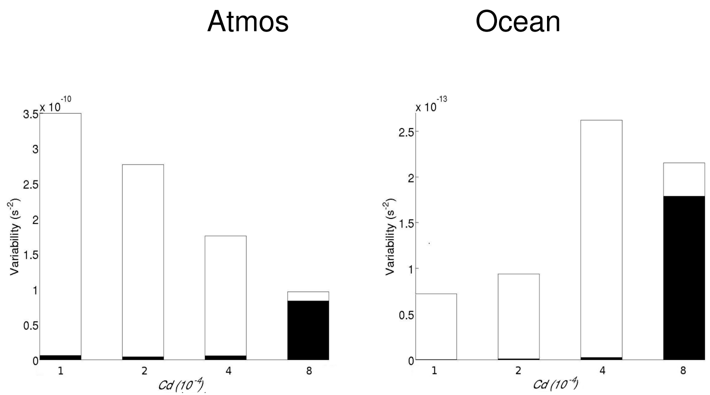
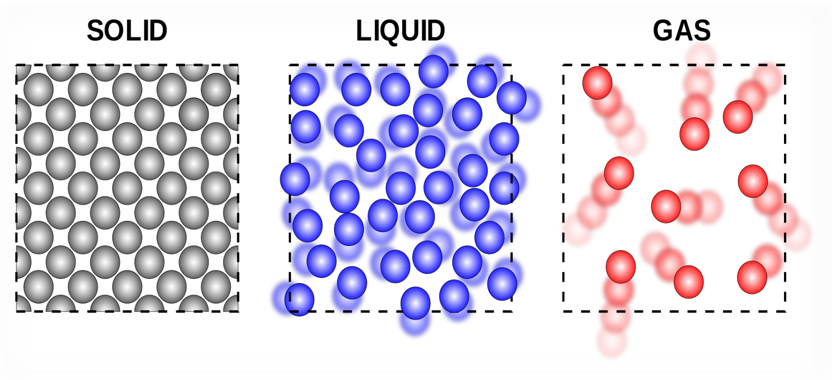


Figure 12.7: Space variability (black) and time variability (white) for four values of the drag coefficient in the atmosphere (right) and in the ocean (left). For the ocean the variability are multiplied by 100 for the three lower drag coefficients.

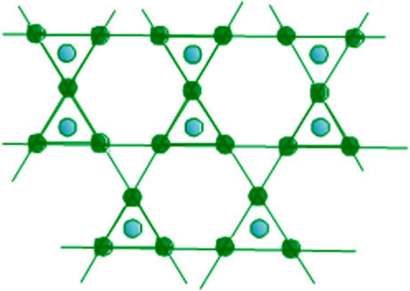
(Moulin & Wirth 2016, BLM 160)

Three Phases

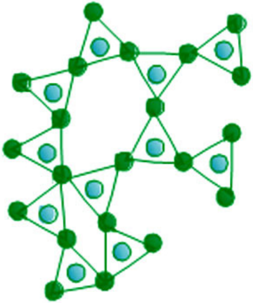


Glassy State

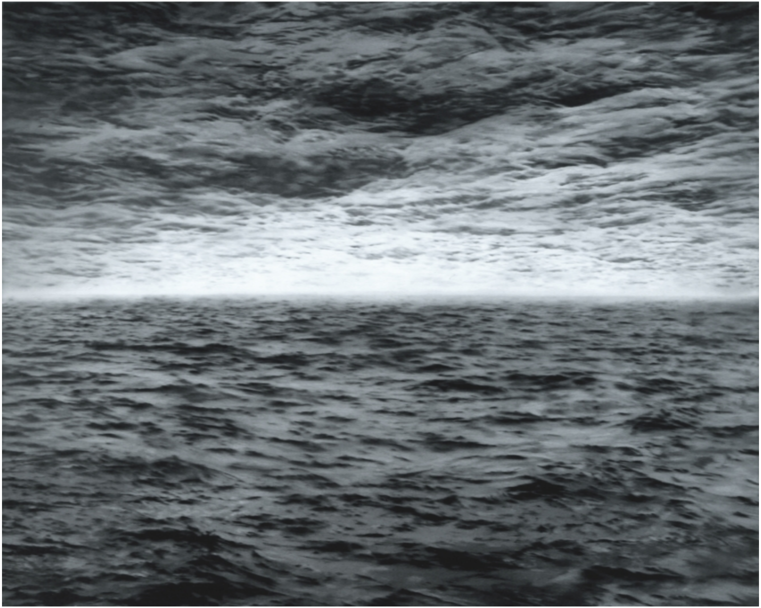
Cristaline solid



Glass



Glasses have the mechanical rigidity of crystals, but the random disordered arrangement of molecules that characterizes liquids.



Seestück (G. Richter)

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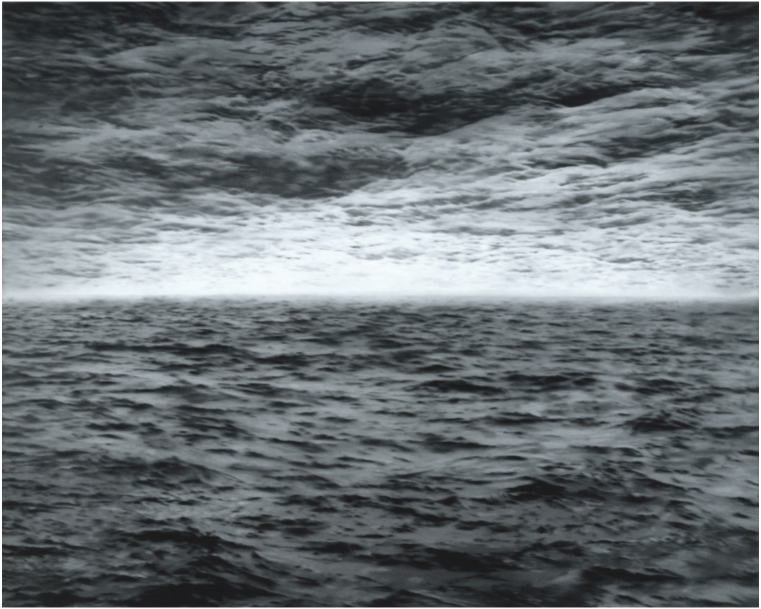
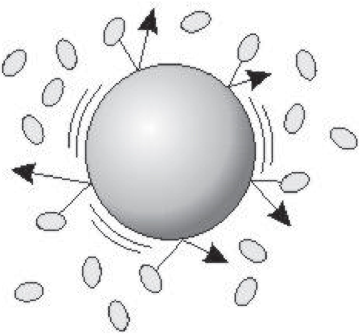
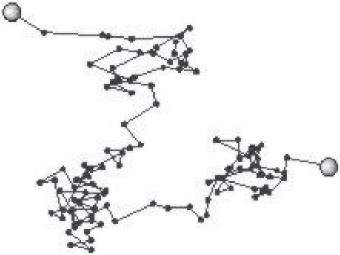


Brownian motion





Einstein relation (1905)

- macroscopic: Stoke's law : $\gamma = \dfrac{6\pi\eta r}{m}$
- microscopic: random walk (1D): D = $\dfrac{\langle x^2 \rangle}{2t} = \dfrac{R}{\gamma^2}$
- equipartition : $\dfrac{k_B T}{m} = \langle u(t)^2 \rangle = \dfrac{R}{\gamma}$

$$D = \frac{k_B T}{\gamma m} = \frac{\mathrm{R}T}{N 6\pi\eta r}$$

Langevin Equation (1908)

$$m\partial_t u(t) = -m\gamma u(t) + F(t)$$

dissipation:	γ	macroscopic	systematic	constant
fluctuationn:	$F(t)$	microscopic	random	$\langle F(t)\rangle = 0$

$$\frac{m}{2}\partial_{tt}x^2 - mu^2 = -\frac{\gamma m}{2}\partial_t x^2 + xF$$

$$\frac{m}{2}\partial_{tt}\langle x^2\rangle - m\langle u^2\rangle = -\frac{m\gamma}{2}\partial_t\langle x^2\rangle + \cancel{\langle xF\rangle}$$

$$\frac{m}{2}\partial_t\langle\partial_t x^2\rangle + \frac{m\gamma}{2}\langle\partial_t x^2\rangle = k_B T$$

$$t \gg \frac{1}{\gamma} \quad \rightarrow \quad \langle\partial_t x^2\rangle = \frac{2k_B T}{m\gamma}$$

Langevin Equation, Itô calculus (1940)

$$u(0) = 0$$

$$u(t) = u(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(t')e^{\gamma t'}dt'$$

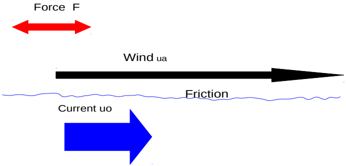
$$\langle u(t)^2\rangle = e^{-2\gamma t}\int_0^t\int_0^t \langle F(t_1)F(t_2)\rangle e^{\gamma(t_1+t_2)}dt_2 dt_1$$

$$\langle F(t_1)F(t_2)\rangle = 2R\delta(t_2 - t_1)$$

Fluctuation dissipation relation:

$$\langle u(t)^2\rangle = \frac{R}{\gamma}$$

Model

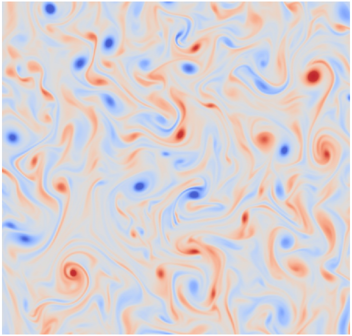


Parameters :

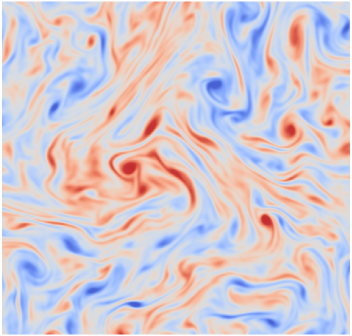
- mass ratio ocean/atmosphere: m
- friction coefficient (nonlinear): c_D

2D Turbulence

Atmos

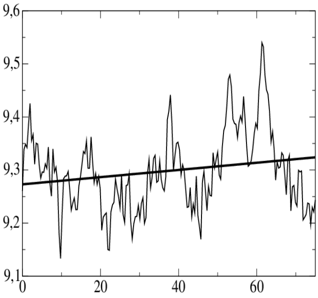


Ocean

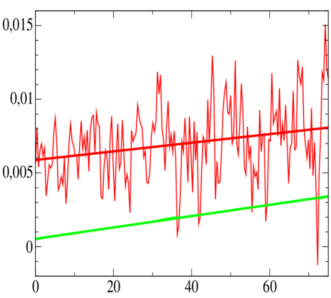


2D Turbulence

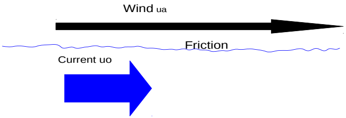
$\langle u_a^2 \rangle_A$



$\langle u_a u_o \rangle_A$, $\langle u_o^2 \rangle_A$



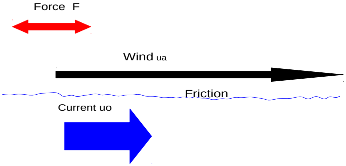
Model



$$\partial_t u_o = -S(u_o - u_a)$$

stat. solution $\leftrightarrow$ 2D turbulence model

Model



$$
\begin{aligned}
\partial_t u_a &= -Sm(u_a - u_o) + F \\
\partial_t u_o &= -S\ \ (u_o - u_a)
\end{aligned}
$$

Linear Local Model

$$
\begin{aligned}
\partial_t u_s &= -SMu_s + F \\
\partial_t u_t &= F
\end{aligned}
$$

$$
\begin{aligned}
u_s(t) &= \int_0^t e^{SM(t'-t)} F(t')dt' \\
u_t(t) &= \int_0^t F(t')dt'
\end{aligned}
$$

$$
\begin{aligned}
u_a(t) &= \frac{1}{M}(u_t + mu_s) = \frac{1}{M}\left(\int_0^t F(t')dt' + m\int_0^t e^{SM(t'-t)} F(t')dt'\right) \\
u_o(t) &= \frac{1}{M}(u_t - u_s) = \frac{1}{M}\left(\int_0^t F(t')dt' - \int_0^t e^{SM(t'-t)} F(t')dt'\right)
\end{aligned}
$$

Linear Local Model : 2nd order moments

$$
\begin{aligned}
\langle u_a^2 \rangle_\Omega &= \frac{R}{M^2}\left(2t + \frac{4m}{SM}(1-e^{-SMt}) + \frac{m^2}{SM}(1-e^{-2SMt})\right)\\
\langle u_o^2 \rangle_\Omega &= \frac{R}{M^2}\left(2t - \frac{4}{SM}(1-e^{-SMt}) + \frac{1}{SM}(1-e^{-2SMt})\right)\\
\langle u_a u_o \rangle_\Omega &= \frac{R}{M^2}\left(2t + \frac{2(m-1)}{SM}(1-e^{-SMt}) - \frac{m}{SM}(1-e^{-2SMt})\right).
\end{aligned}
$$

For $t \gg (SM)^{-1}$:

$$
\begin{aligned}
\langle (u_a - u_o)^2 \rangle_\Omega &= \frac{R}{SM}\\
\langle u_a^2 - u_o^2 \rangle_\Omega &= \frac{R(M+2)}{SM^2}\\
\langle u_a u_o - u_o^2 \rangle_\Omega &= \frac{R}{SM^2}
\end{aligned}
$$

Fluctuation Dissipation Relation (FDR)

$$\frac{1}{2}\partial_t \langle u_o^2 \rangle_\Omega = S\langle u_a u_o - u_o^2 \rangle_\Omega = \frac{R(1-e^{-SMt})^2}{M^2}$$

For $t \gg (SM)^{-1}$:

$$\frac{R}{M^2} = \frac{SR}{M^2}\left(2t + \frac{m-2}{SM} - 2t + \frac{3}{SM}\right)$$

Quadratic Local Model

$$\begin{aligned}\partial_t \mathbf{u}_a &= - \quad \tilde{S} m |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \\ \partial_t \mathbf{u}_o &= \quad \tilde{S} \; |\mathbf{u}_s| \mathbf{u}_s\end{aligned}$$

with $\mathbf{u}_s = \mathbf{u}_a - \mathbf{u}_o$, $\mathbf{u}_t = \mathbf{u}_a + m\mathbf{u}_o$.

$$\begin{aligned}\partial_t \mathbf{u}_s &= -\tilde{S} M |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \\ \partial_t \mathbf{u}_t &= \mathbf{F}\end{aligned}$$

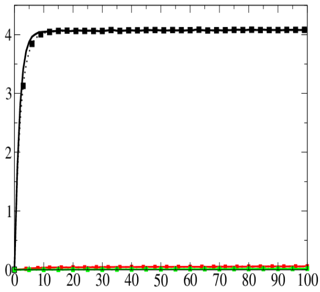
Linear Langevin eq. with eddy friction:

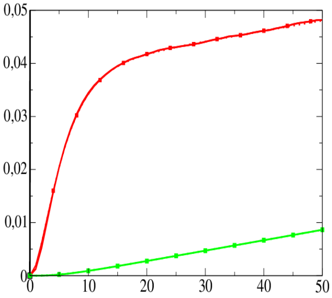
$$\frac{S_{\text{eddy}}}{\tilde{S}} = \frac{\langle (\mathbf{u}_s^2)^{3/2} \rangle}{\langle \mathbf{u}_s^2 \rangle^{3/2}} \langle (\mathbf{u}_s^2)^{1/2} \rangle = \left(\frac{\mu^2 2R}{\tilde{S} M} \right)^{1/3} .$$

$$\mu_{\text{Gaussian}} = \frac{\langle (\mathbf{u}_s^2)^{3/2} \rangle}{\langle \mathbf{u}_s^2 \rangle^{3/2}} = \frac{3\sqrt{\pi}}{4} \approx 1.3293404.$$

Lin. vs. Quadratic Langevin eq.

$$\langle u_a^2 \rangle_A, \langle u_o^2 \rangle_A, \langle u_a u_o \rangle_A$$





$$\mu = \frac{2}{3}\frac{\Gamma(2/3)}{\Gamma(4/3)} \approx 1.2449; (Gaussian) = \frac{3\sqrt{\pi}}{4} \approx 1.329$$

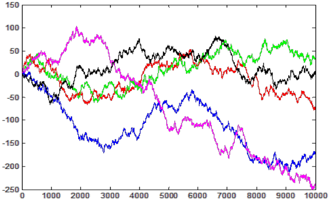
Stochastic differential equation:

Integrating many independent realisation:

$$\partial_t u = F(u, \omega) \quad \text{with,} \quad \omega \in \Omega$$

$\rightarrow$ measure moments :

$$\langle u^n \rangle_\Omega, \qquad \langle f(u) \rangle_\Omega$$



("Lagrangian approach")

Fokker-Planck equation:

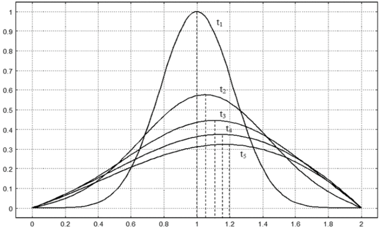
Obtain PDE for the time evolution of the pdf:

$$\partial_t P(u,t) = \partial_u \left(a(u)P(u) + \frac{1}{2}\partial_u \left[b(u)P(u) \right] \right)$$

$\rightarrow$ solve equation if possible and obtain moments by integration:

$$\langle u^n \rangle = \int u^n dP, \qquad \langle f(u) \rangle = \int f(u) dP$$

(“Eulerian approach”)



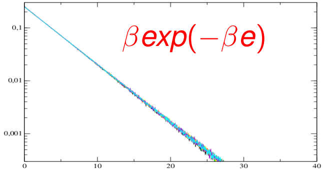
Linear model: SDE $\leftrightarrow$ Fokker-Planck equation:

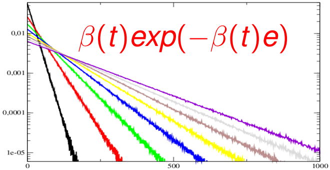
SDE:

$$\begin{aligned} \partial_t \mathbf{u}_s &= -SM\mathbf{u}_s + F \\ \partial_t \mathbf{u}_t &= F \end{aligned}$$

Fokker-Planck

$$\begin{aligned} \partial_t P_s &= \nabla_{uv} \cdot \left[SM\mathbf{u}_s P_s + \frac{1}{2} \nabla_{uv} P_s \right] \\ \partial_t P_t &= \frac{1}{2} \nabla_{uv} \cdot \nabla_{uv} P_t \end{aligned}$$





Non-linear model: SDE $\leftrightarrow$ Fokker-Planck equation:

SDE:

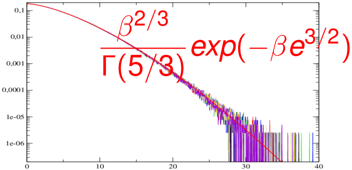
$$\partial_t \mathbf{u}_s = -\tilde{S}M|\mathbf{u}_s|\mathbf{u}_s + \mathbf{F} \quad (1)$$

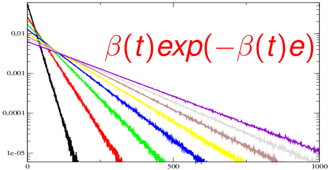
$$\partial_t \mathbf{u}_t = \mathbf{F} \quad (2)$$

Fokker-Planck

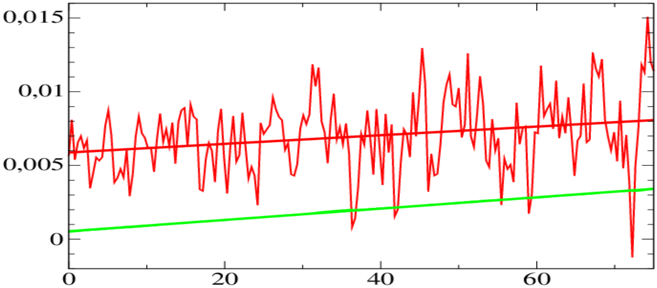
$$\partial_t P_s = \nabla_{uv} \cdot \left[\tilde{S}M\mathbf{u}_s u_s P_s + \frac{\nu}{2}\nabla_{uv} P_s\right]$$

$$\partial_t P_t = \frac{\nu}{2}\nabla_{uv} \cdot \nabla_{uv} P_s$$





FDR 2D : $\langle \mathbf{u}_o^2 \rangle$, $\langle u_a u_o \rangle$



$$\frac{1}{2}\partial_t \langle u_o^2 \rangle_A = S \langle u_a u_o - u_o^2 \rangle_A$$

$$\tilde{S}_{\mathrm{num}} = \frac{\partial_t \langle \mathbf{u}_o^2 \rangle}{2\mu_{\mathrm{Gauss}} \sqrt{\langle (\mathbf{u}_a - \mathbf{u}_o)^2 \rangle} \langle (\mathbf{u}_a \mathbf{u}_o - \mathbf{u}_o^2) \rangle}$$

$$\frac{\tilde{S}_{\mathrm{num}}}{\tilde{S}} = 0.9$$

Conclusions

* The ocean subject to atmospheric forcing obeys a fluctuation dissipation relation.
* Local models (linear and quadratic) can be solved analytically.
* Some of the results from local models can be transposed to fully 2D turbulence models.
* Just at the beginning.

- Dissipation of non-resolved dynamics is included in models (atmosphere, ocean climate, ...) but not the fluctuations. However, fluctuation-dissipation-relations hold at all levels of the dynamics.
- Consider truely non equilibrium processes (beyond: spin-up, spin-down)