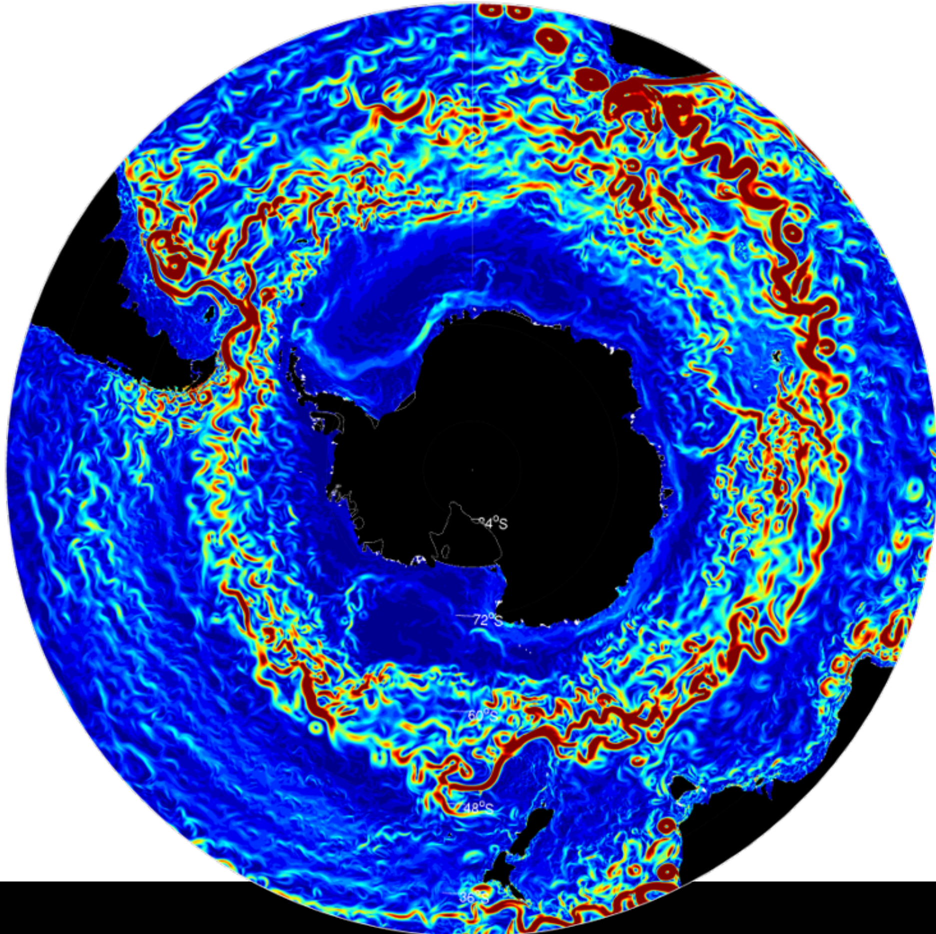
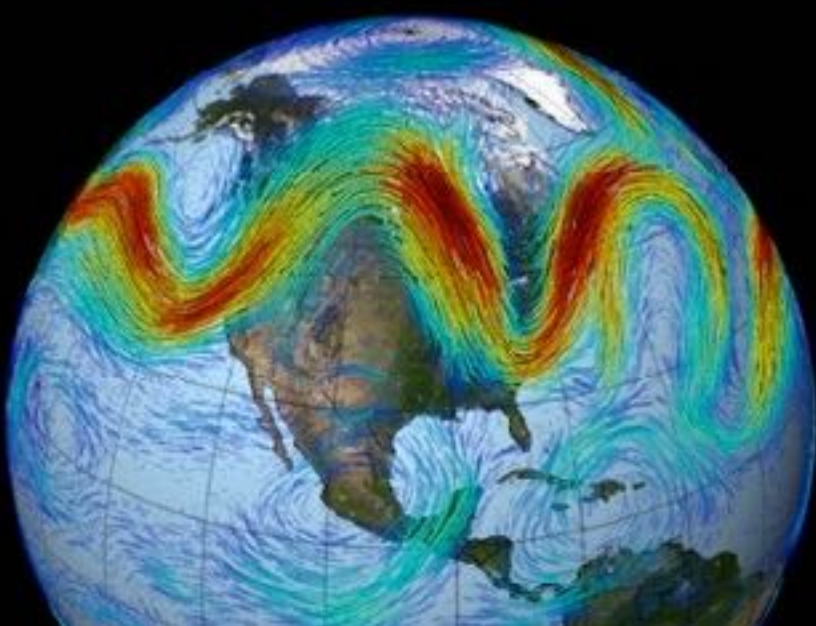
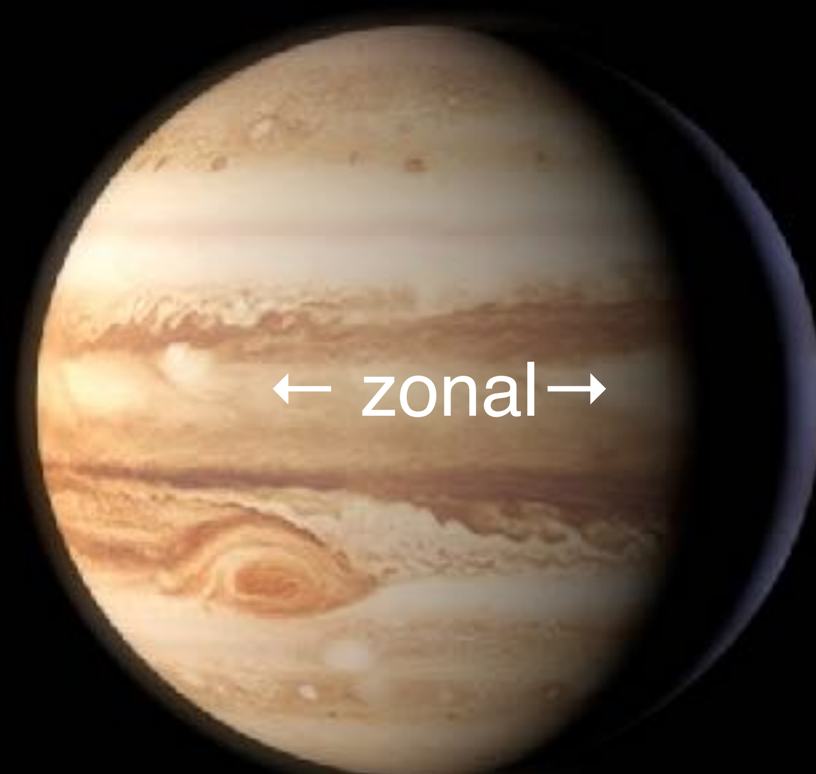
Geostrophic turbulence







Geostrophic turbulence is 2D turbulence with additional complications

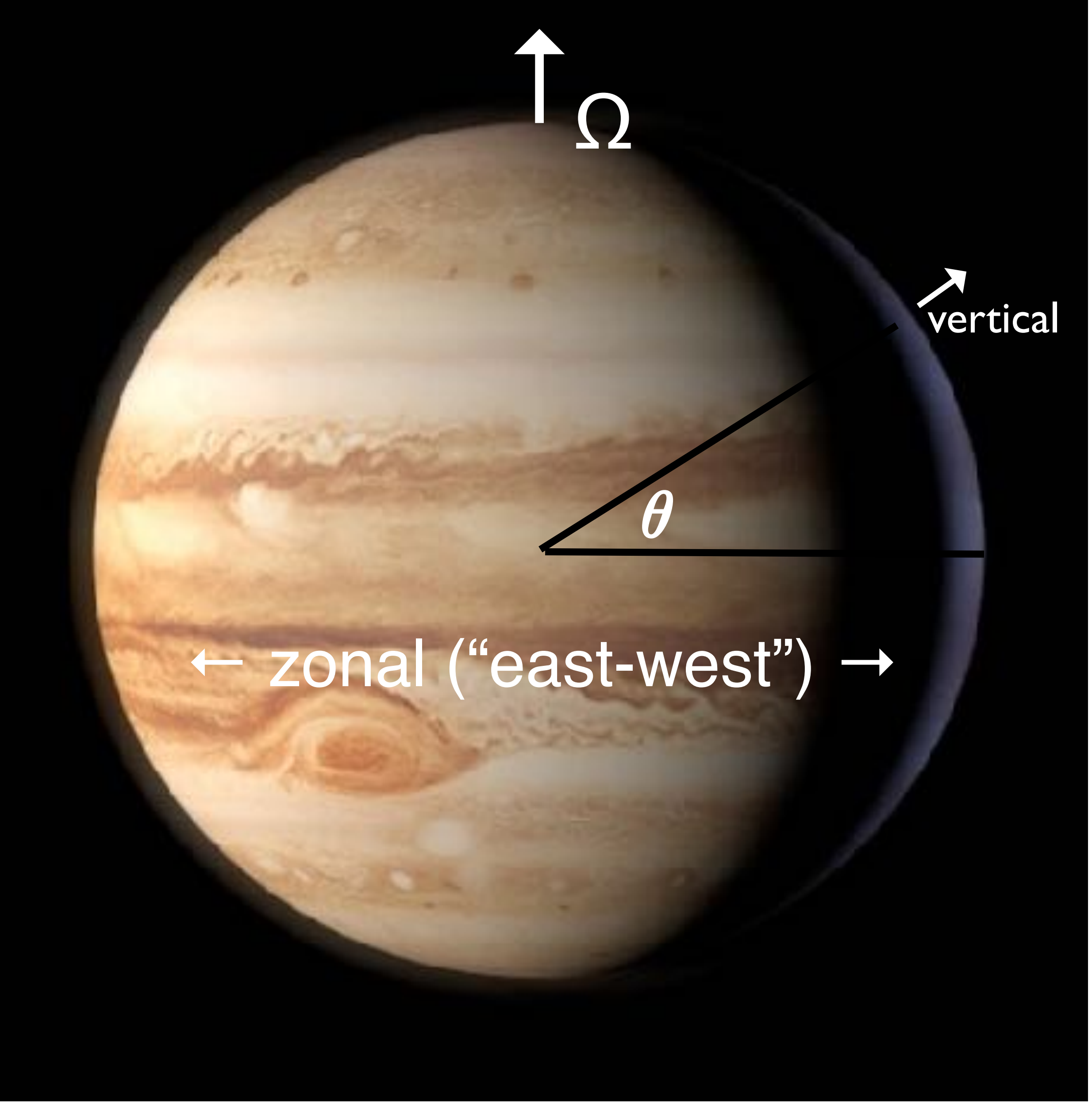
The 2Dness is justified by rapid rotation.

The beta effect, "zonation" and "zonostrophic instability"

Non-uniform layer depth due to topography

Stratification, AKA "baroclinicity" and baroclinic instability

Coupling to internal gravity waves and other unbalanced motions.



The beta-plane

In a thin, stratified spherical layer only the local-vertical component of Ω is important.

$$f = f_0 + \beta y$$

$$2\Omega_{\mathrm{vert}} = 2\Omega \times \sin\theta \approx \underbrace{2\Omega \sin\theta_0}_{f_0} + \underbrace{2\Omega R^{-1}\cos\theta_0 \times R(\theta - \theta_0)}_{\beta \times y}$$

The "standard" barotropic model

Material conservation of QGPV

$$q_t + uq_x + vq_y = \text{forcing} + \text{dissipation}$$

$$(u, v) = (-\psi_y, \psi_x)$$

$$q = \underbrace{\psi_{xx} + \psi_{yy}}_{\zeta} + \beta y$$

If β=0 we have 2D turbulence.
With non-zero β we can study the cross-over between waves and turbulence.

The linearized equation, with no F and D, has Rossby wave solutions.

$$(\psi_{xx} + \psi_{yy})_t + \beta\psi_x = 0$$

A linear interlude

$$(\psi_{xx} + \psi_{yy})_t + \beta \psi_x = 0$$

Rossby Waves

$$(\psi_{xx} + \psi_{yy})_t + \beta\psi_x = 0$$

The dispersion relation

$$\psi = \exp\left[\mathrm{i}(kx + ly - \sigma t)\right]$$

$$\Rightarrow \sigma = -\frac{\beta k}{k^2 + l^2}$$

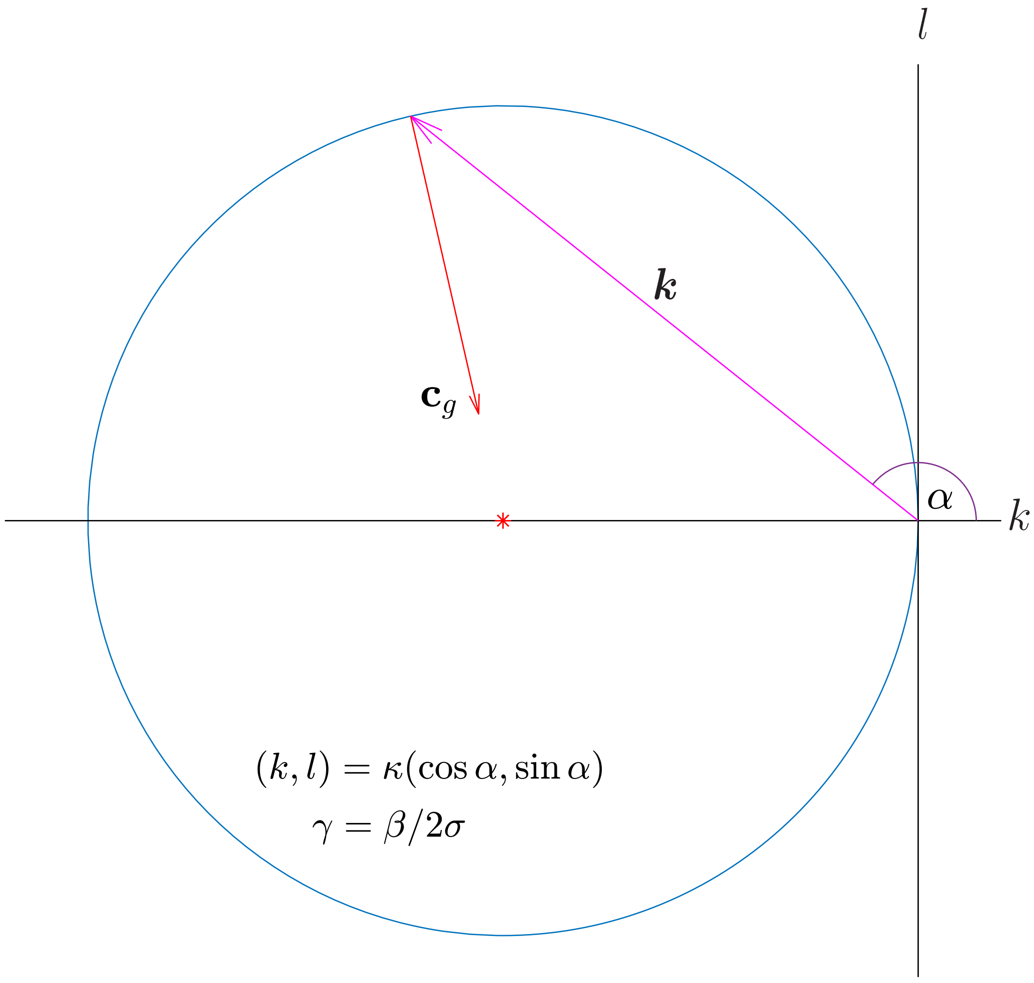
The dispersion circle

$$\left(k^2 + \frac{\beta}{2\sigma}\right)^2 + l^2 = \left(\frac{\beta}{2\sigma}\right)^2$$

The group velocity

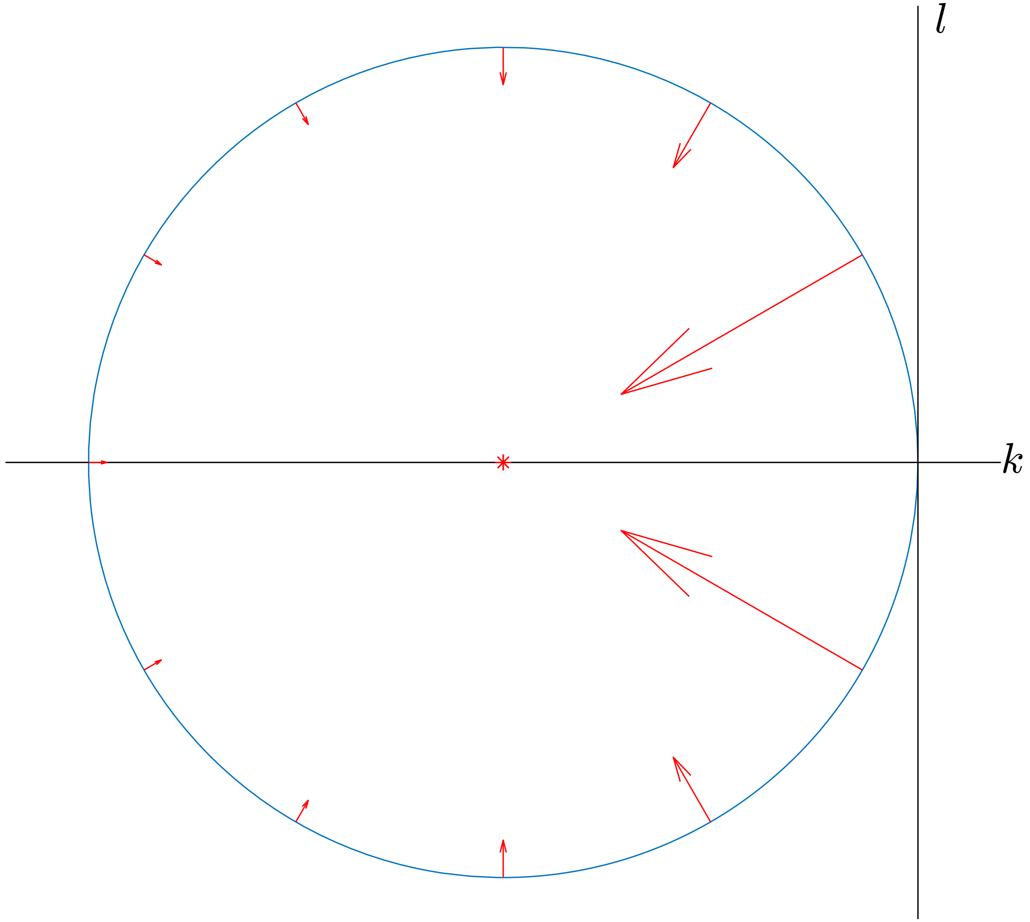
$$\boldsymbol{c}_{\mathrm{g}} = (\sigma_k \, , \sigma_l)$$

$$= \frac{\beta}{\kappa^2}\,(\cos 2\alpha \, , \sin 2\alpha)$$



$$\sigma = -\frac{\beta k}{k^2 + l^2}$$

Very different spatial scales have the same Rossby wave frequency.



And the amplitude of the group velocity is bigger for longer waves.

$$|\boldsymbol{c}_{\mathrm{g}}| = \frac{\beta}{k^2 + l^2}$$

A Green's Function example

Oscillatory forcing

$$(G_{xx} + G_{yy})_t + \beta G_x = \delta(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}$$

An inspired guess produces the Helmholtz equation…

$$G(x, y, t) = \frac{\exp\left[-\mathrm{i}(\omega t + \frac{\beta x}{2\omega})\right]}{-\mathrm{i}\omega}\mathcal{G}(x, y)$$

$$\mathcal{G}_{xx} + \mathcal{G}_{yy} + \left(\frac{\beta}{2\omega}\right)^2 \mathcal{G} = \delta(\boldsymbol{x})$$

The solution — one must apply the radiation condition.

$$G = \frac{\exp\left[-\mathrm{i}(\omega t + \gamma x)\right]}{-\mathrm{i}\omega} H_0^{(2)}(\gamma r)$$

$J_0 - \mathrm{i}Y_0$

The radius of the dispersion circle is:

$$\gamma = \frac{\beta}{2\omega}$$

Haidvogel & Rhines (1983)

The far-field wave crests are parabolas and the phase is

$$p = -\gamma(r + x)$$

Local wavenumber are

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} p = -\gamma \begin{pmatrix} 1 + \cos\theta \\ \sin\theta \end{pmatrix}$$

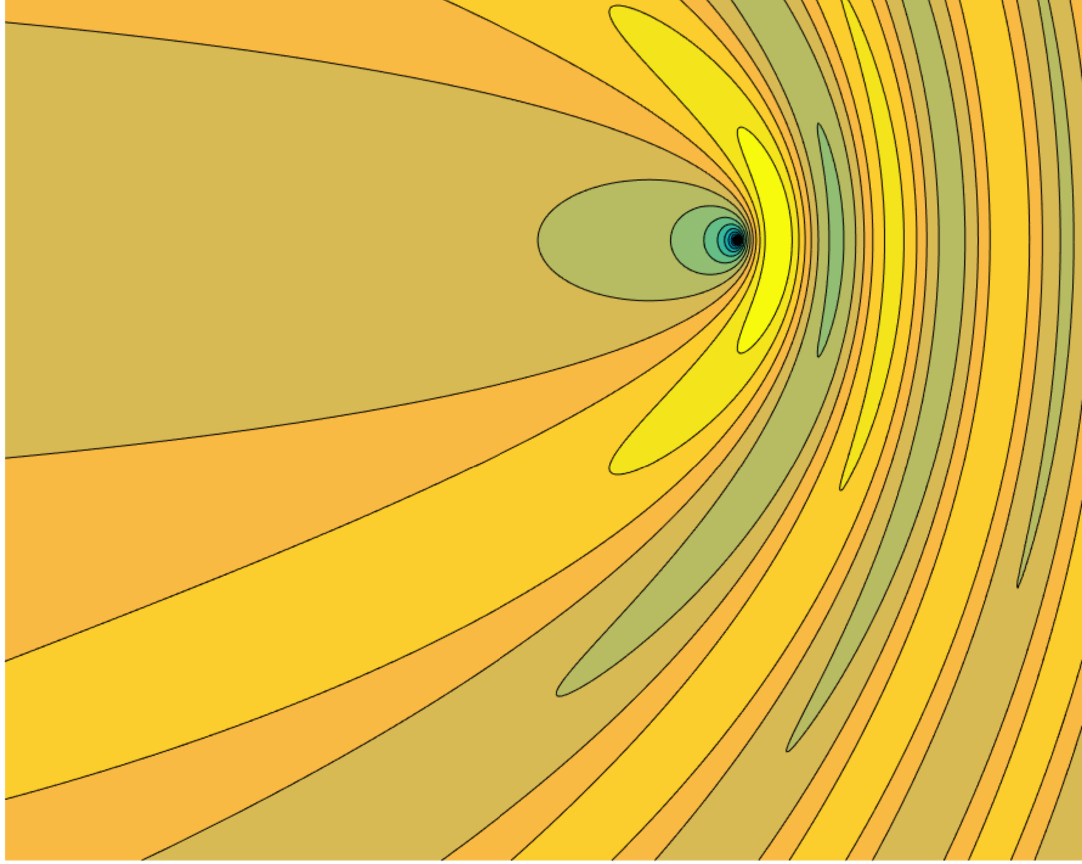
A sanity check

$$k^2 + l^2 = 2\gamma^2(1 + \cos\theta) = -\frac{\beta k}{\omega} \quad ✓$$

and $\tan\alpha = \dfrac{l}{k} = \dfrac{\sin\theta}{1 + \cos\theta} = \tan\dfrac{\theta}{2}$ ✓

Waves radiated in the direction θ have the right (= physically intuitive) group velocity

Note how the different spatial scales in this mono frequency solution are sorted by the direction of the group velocity.



$$G \approx \frac{\mathrm{i}}{\omega}\sqrt{\frac{2}{\pi\gamma r}}\,\mathrm{e}^{-\mathrm{i}\gamma(r+x) - \mathrm{i}\omega t - \frac{\pi}{4}},$$

provided $\dfrac{\beta r}{2\omega} \gg 1$

$$(x\,,\,y) = r(\cos\theta\,,\,\sin\theta)$$

$$\gamma = \frac{\beta}{2\omega}$$

The energy density

$$E = \tfrac{1}{2}|\boldsymbol{\nabla} G|^2 \approx \frac{1}{\omega^2}\frac{4\gamma}{\pi r}(1+\cos\theta)$$

(Note zero energy density
due west of the source!)

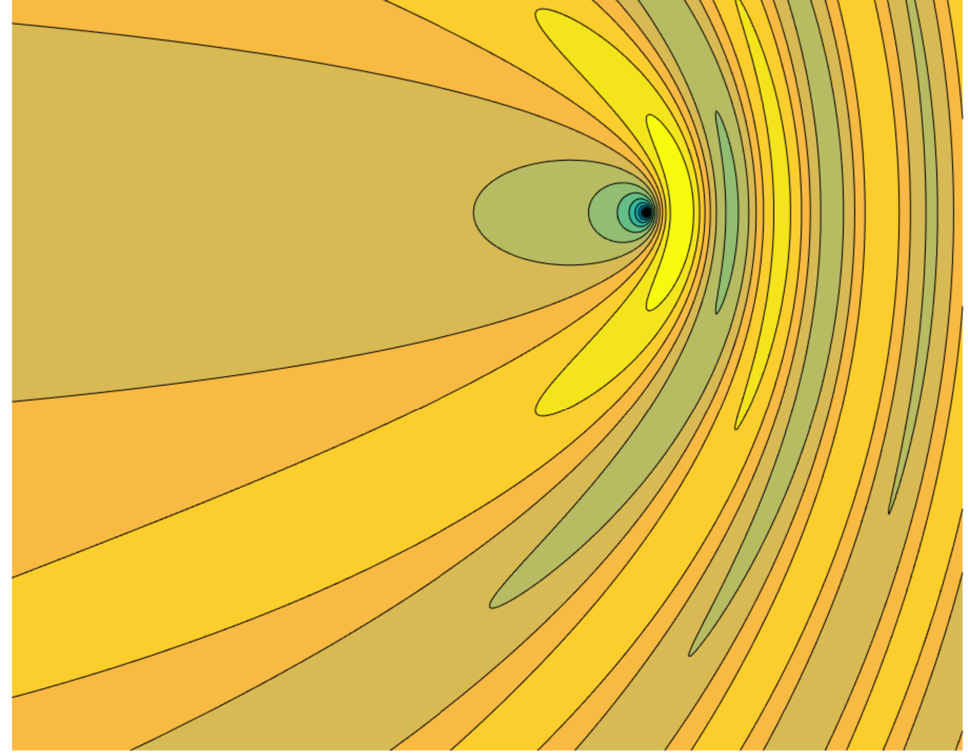
But the energy flux is isotropic:

$$\boldsymbol{F} = \boldsymbol{c}_{\mathrm{g}} E = \frac{2}{\omega\pi}\frac{\hat{\boldsymbol{r}}}{r}$$

$$|\boldsymbol{c}_{\mathrm{g}}| = \frac{\beta}{k^2 + l^2}$$

(Note infinite group velocity
due west of the source.)

Is isotropic radiation obvious?
Hummmm



(Haidvogel & Rhines 1983)

$$(G_{xx} + G_{yy})_t + \beta G_x = \delta(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}$$

The Green's function radiates isotropically because it contains all spatial Fourier components with equal weight. If the source of waves has a limited mix of spatial components, say with a Gaussian forcing pattern, $\exp[-r^2/a^2 - i\omega_0 t]$, strong anisotropy then occurs. The familiar argument is that the locus (Figure 8) of possible wave vectors at a fixed frequency, ω_0, determines through its normals the directions of the rays. The far field response is simply the product of the (perhaps) symmetrical forcing spectrum with this unsymmetrical locus, or

$$\psi \simeq G(x, y)\, e^{-i\omega_0 t}\, P(\theta), \qquad Br \gg 1, \tag{3.4}$$

End of the linear interlude

Go back to the nonlinear problem

$$\zeta_t + \psi_x \zeta_y - \psi_y \zeta_x + \beta \psi_x = 0$$

$$\zeta = \psi_{xx} + \psi_{yy}$$

Solution of the IVP

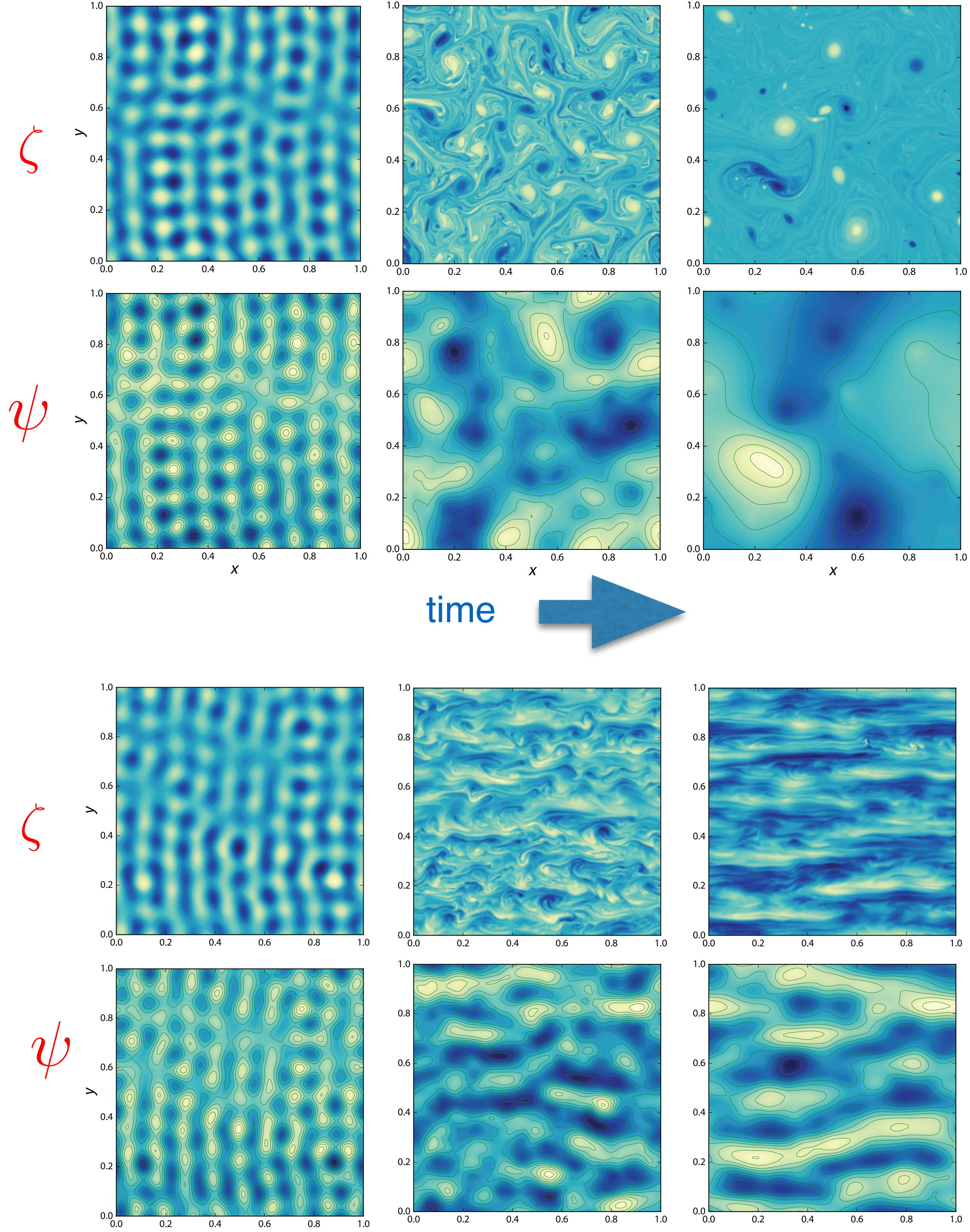
$\beta = 0$

Plain old 2D turbulence —
vortex gas etc.

$\beta \neq 0$

Instead of vortices, we see formation of zonal jets, or *zonation*.

(Vallis 2017)



The Rhines length

$$\ell_{\text{Rhines}} = \sqrt{\frac{U}{\beta}}$$

The Rhines length

$$\zeta_t + J(\psi, \zeta) + \beta\psi_x = \nu\nabla^2\zeta$$

Beta makes no difference to the conservation laws

$$\beta\langle\psi\psi_x\rangle = \beta\langle\zeta\psi_x\rangle = 0$$

So Energy is still robustly conserved.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tfrac{1}{2}|\boldsymbol{\nabla}\psi|^2\rangle = -\nu\langle\zeta^2\rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tfrac{1}{2}\zeta^2\rangle = -\nu\langle|\boldsymbol{\nabla}\zeta|^2\rangle$$

But now we have two dimensional parameters.

(Rhines 1975)

$$U \stackrel{\mathrm{def}}{=} \sqrt{\frac{1}{2}\langle|\boldsymbol{\nabla}\psi|^2\rangle}$$

$$\ell_{\mathrm{Rhines}} = \sqrt{\frac{U}{\beta}}$$

The Rhines length is the emergent scale of the jet spacing.

The Vallis & Maltrud dumbbell

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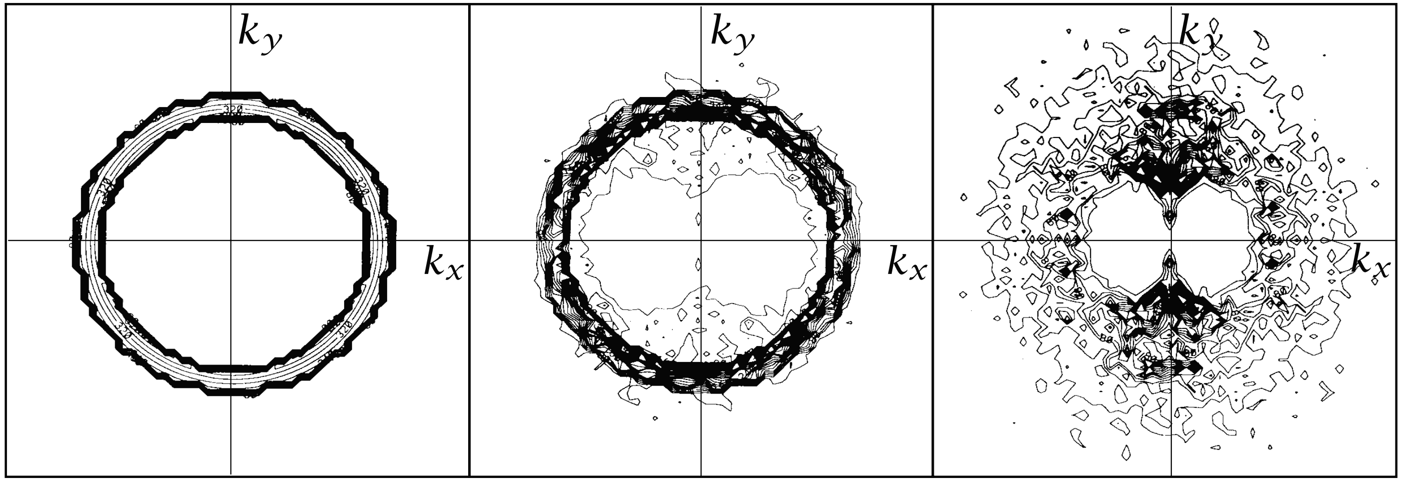
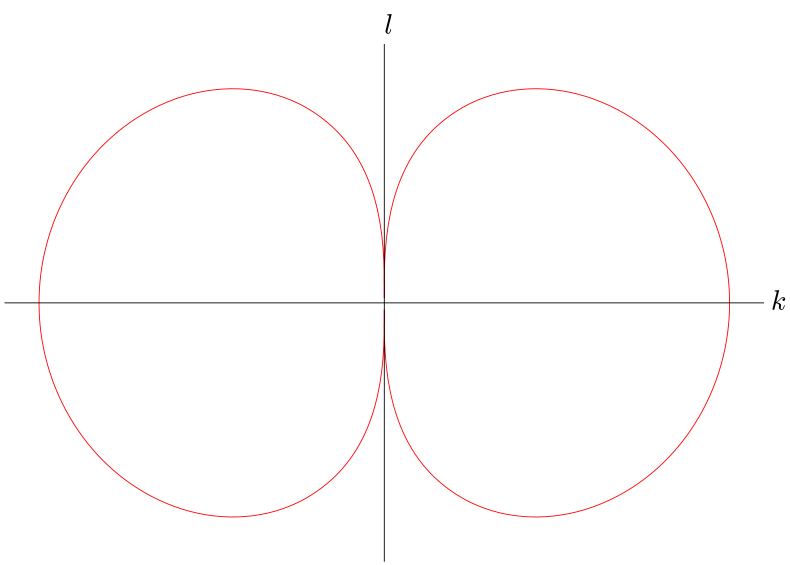
The Vallis & Maltrud dumbbell



Evolution of the energy spectrum. The initial spectrum is isotropic. The transfer of energy to large scales is impeded by the beta effect.

$$U\sqrt{k^2+l^2} \sim \frac{\beta k}{k^2+l^2} \qquad \Rightarrow \qquad \sqrt{k^2+l^2} = \sqrt{\frac{\beta}{U}|\cos\alpha|}$$

"Within the dumbbell characteristic Rossby wave times are shorter than the turbulent turnover times. This inhibits transfer of energy from the turbulent regime because efficient forcing of a wavelike mode will be achieved only when the forcing frequency is comparable to the natural frequency."



Now the forced
β-plane problem

Forced beta-plane turbulence (and Ekman drag)

(Vallis & Maltrud 1991, 1993)

$$\zeta_t + u\zeta_x + v\zeta_y + \beta v = \xi - \mu\zeta + \nu\nabla^2\zeta$$

$$u = -\psi_y\,, \qquad v = \psi_x \qquad \zeta = \psi_{xx} + \psi_{yy}$$

$$\xi(x, y, t) = \text{specified forcing}$$

The forcing models smaller scale processes (baroclinic eddies, convection, baroclinic instability).

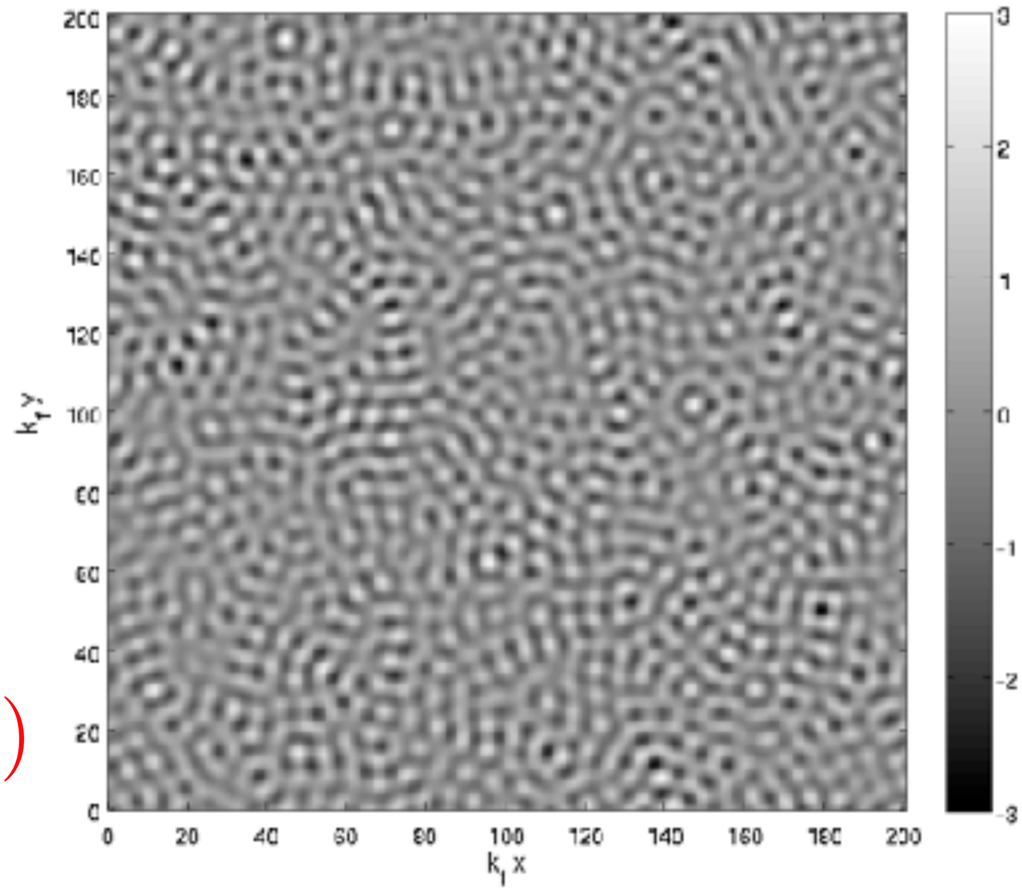
A popular (but not universal) assumption is that the forcing is characterized by its energy injection rate ε and nothing else is important. For example, the length scale of the forcing is irrelevant — provided it is small enough.

The most popular forcing is "white noise"

Used by Ted Shepherd last week?

Lilly 1969

$\xi(x, y, t)$



$k_f L = 32$

forcing wavenumber — should be irrelevant?

Homogeneous isotropic, spectrally narrow-band, rapidly decorrelating, small scale etc.

$$\overline{\xi(x_1, y_1, t_1)\xi(x_2, y_2, t_2)} = \delta(t_1 - t_2)\,\Xi(r)$$

The zonal average

(average over the "homogeneous direction")

Definition of zonal average

$$\overline{\text{anything}} = \tfrac{1}{L}\int_0^L \text{anything}\,\mathrm{d}x$$

Quantities with zero zonal average

$$\overline{(\text{anything})_x} = 0 \qquad \text{and} \qquad \bar{\xi} = 0$$

$$\bar{v} = \overline{\psi_x} = 0$$

The zonal mean flow

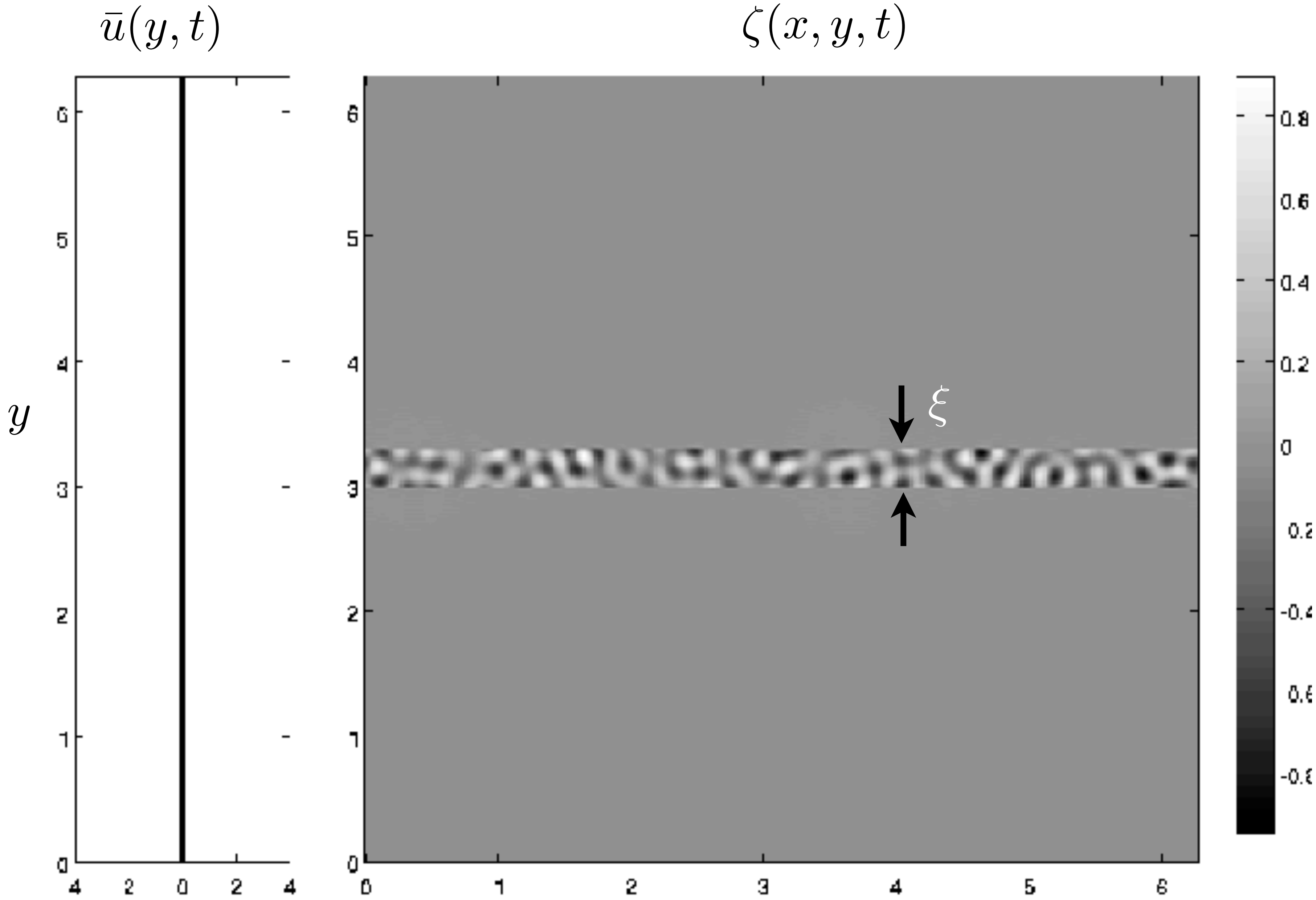
$$\bar{u}(y,t) = \frac{1}{L}\int_0^L u(x,y,t)\,\mathrm{d}x$$

Reynolds decomposition

$$u = \bar{u} + u'$$

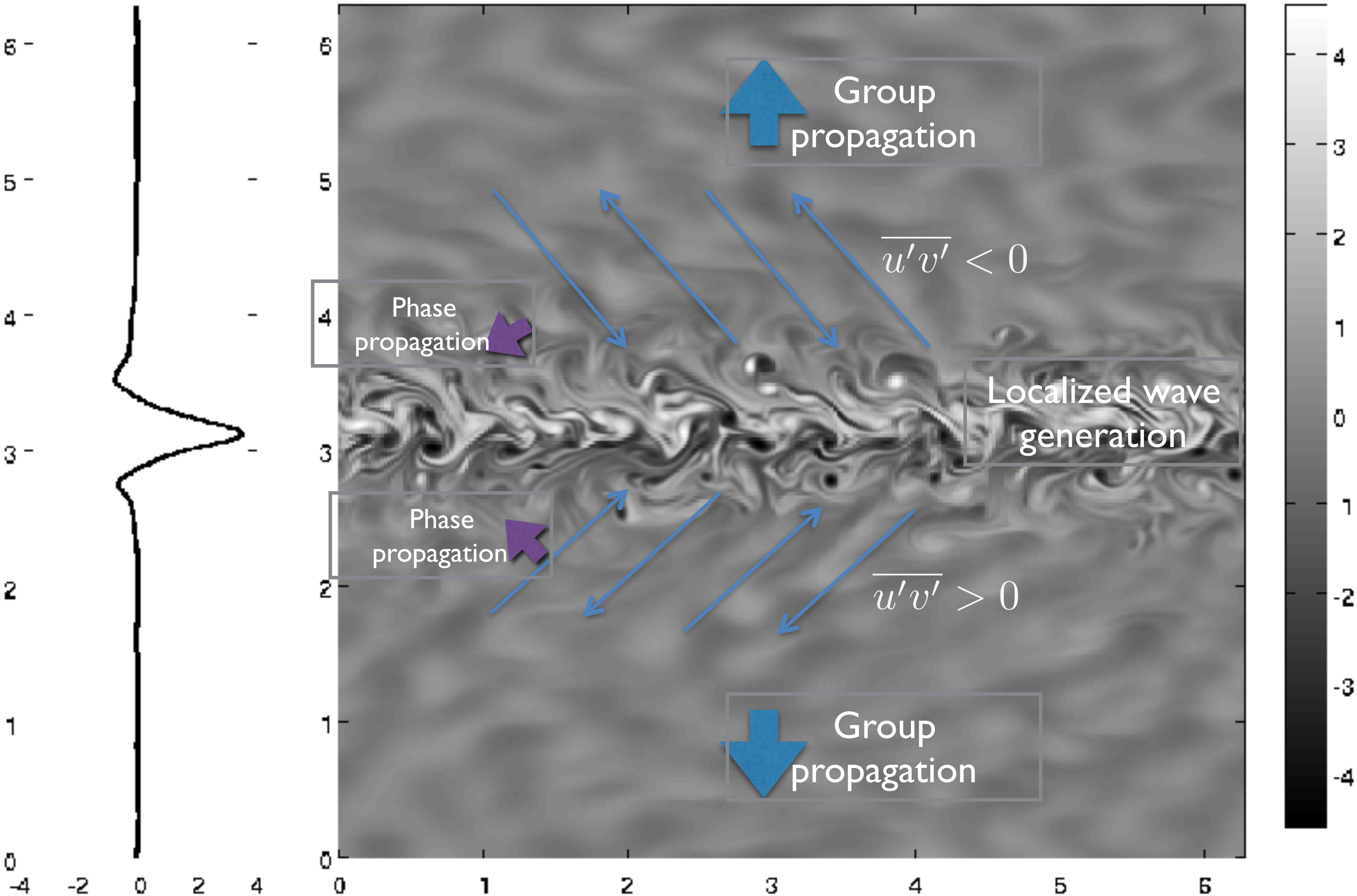
The forced strip — momentum is unmixed

(Whitehead, McEwan, Thompson, Plumb and Rhines)



$$\int_{-\infty}^{\infty} \bar{u}(y,t)\,\mathrm{d}y = 0$$

What did we just see?



If pressed, we can substantiate this scenario in detail…

An amplitude expansion

$$\xi = \epsilon \xi_1$$

$$\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots$$

Forced Rossby waves at leading order

$$\left[\partial_t \left(\partial_x^2 + \partial_y^2\right) + \beta \partial_x + \mu \left(\partial_x^2 + \partial_y^2\right)\right] \psi_1 = \xi_1$$

Now calculate the Reynolds stress and use the zonal-mean momentum equation.

$$\mu \bar{u}_2 = -\partial_y \left(\overline{u_1 v_1}\right)$$

Instead, let's examine the eddy enstrophy power integral.

Deductions from enstrophy

The eddy PV equation

$$\zeta'_t + \bar{u}\zeta'_x + (\beta - \bar{u}_{yy})\, v' + \boldsymbol{\nabla}\cdot\left(\boldsymbol{u}'\zeta' - \overline{\boldsymbol{u}'\zeta'}\right) = -\mu\zeta' + \nu\nabla^2\zeta'$$

The eddy enstrophy equation

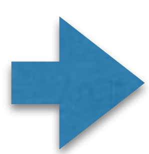
$$\left(\tfrac{1}{2}\overline{\zeta'^2}\right)_t + (\beta - \bar{u}_{yy})\,\overline{v'\zeta'} + \left(\tfrac{1}{2}\overline{v'\zeta'^2}\right)_y = \overline{\xi\zeta} - \mu\overline{\zeta'^2} - \nu\overline{|\nabla\zeta'|^2} + \nu\left(\tfrac{1}{2}\overline{\zeta'^2}\right)_{yy}$$

With weak non-linearity

$$\beta\,\overline{v'\zeta'} \approx \overline{\xi\zeta} - \mu\overline{\zeta'^2} - \nu\overline{|\nabla\zeta'|^2}$$

Recall Taylor's identity

$$\mu\bar{u} = \overline{v'\zeta'} = -\left(\overline{u'v'}\right)_y$$



$$\bar{u} = \frac{\overline{\xi\zeta'} - \mu\overline{\zeta'^2} - \nu\overline{|\boldsymbol{\nabla}\zeta'|^2}}{\mu\beta}$$

The ZMF is the difference between enstrophy production and enstrophy dissipation.
There is westward flow in unforced regions

Prandtl versus Taylor

$$\overline{v'\zeta'} = -\kappa_T \bar{q}_y \quad \text{versus} \quad \overline{u'v'} = -\nu_P \bar{u}_y$$

But $\bar{q}_y = \beta - \bar{u}_{yy} \approx \beta \qquad \therefore \; \overline{v'\zeta'} = -\kappa_T \beta$

The zonal momentum equation is

$$\bar{u}_t = \overline{v'\zeta'}$$

$$\approx -\kappa_T \beta$$

momentum conservation ⟶

$$\int \kappa_T \, \mathrm{d}y = 0$$

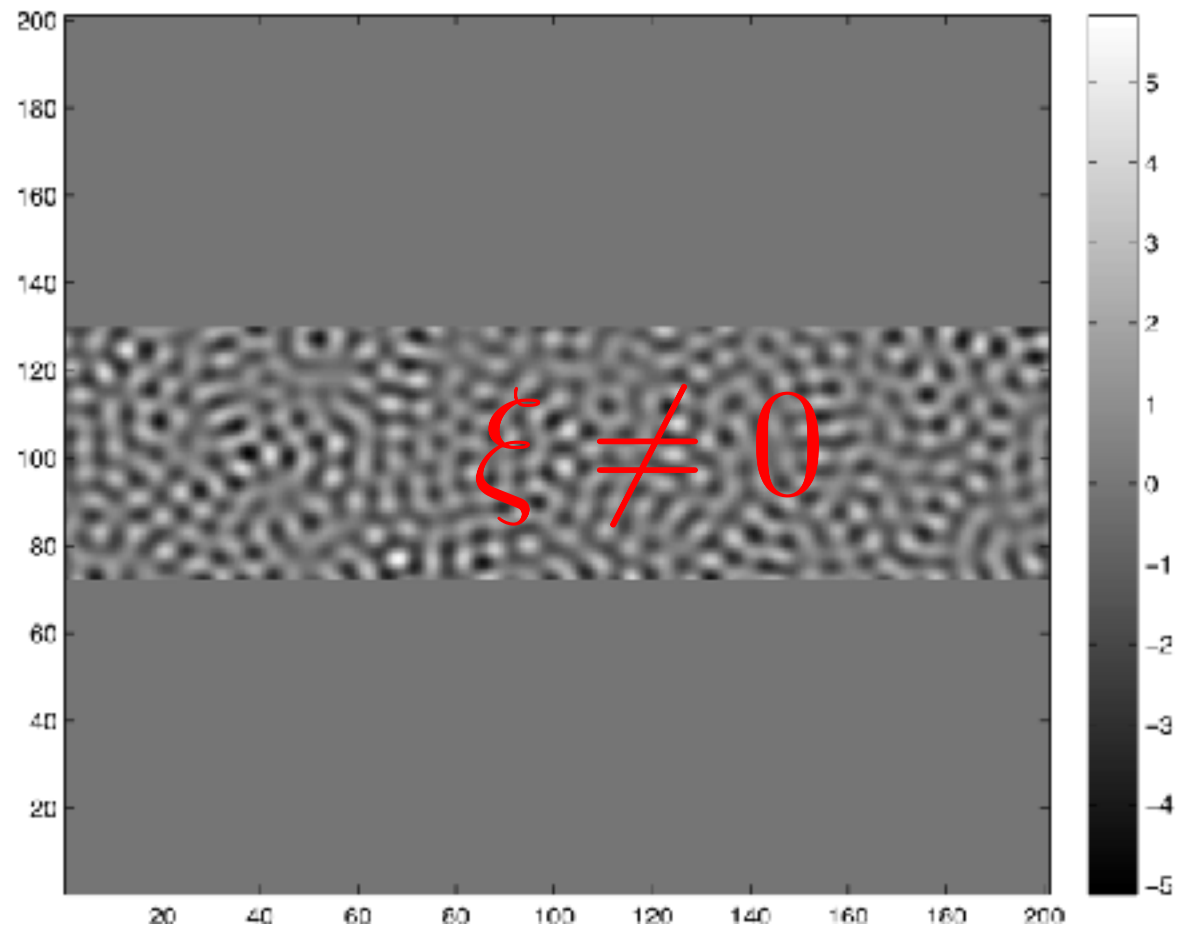
The Taylor PV diffusivity cannot be positive definite — that's not good. The Prandtl eddy viscosity does not have this issue.

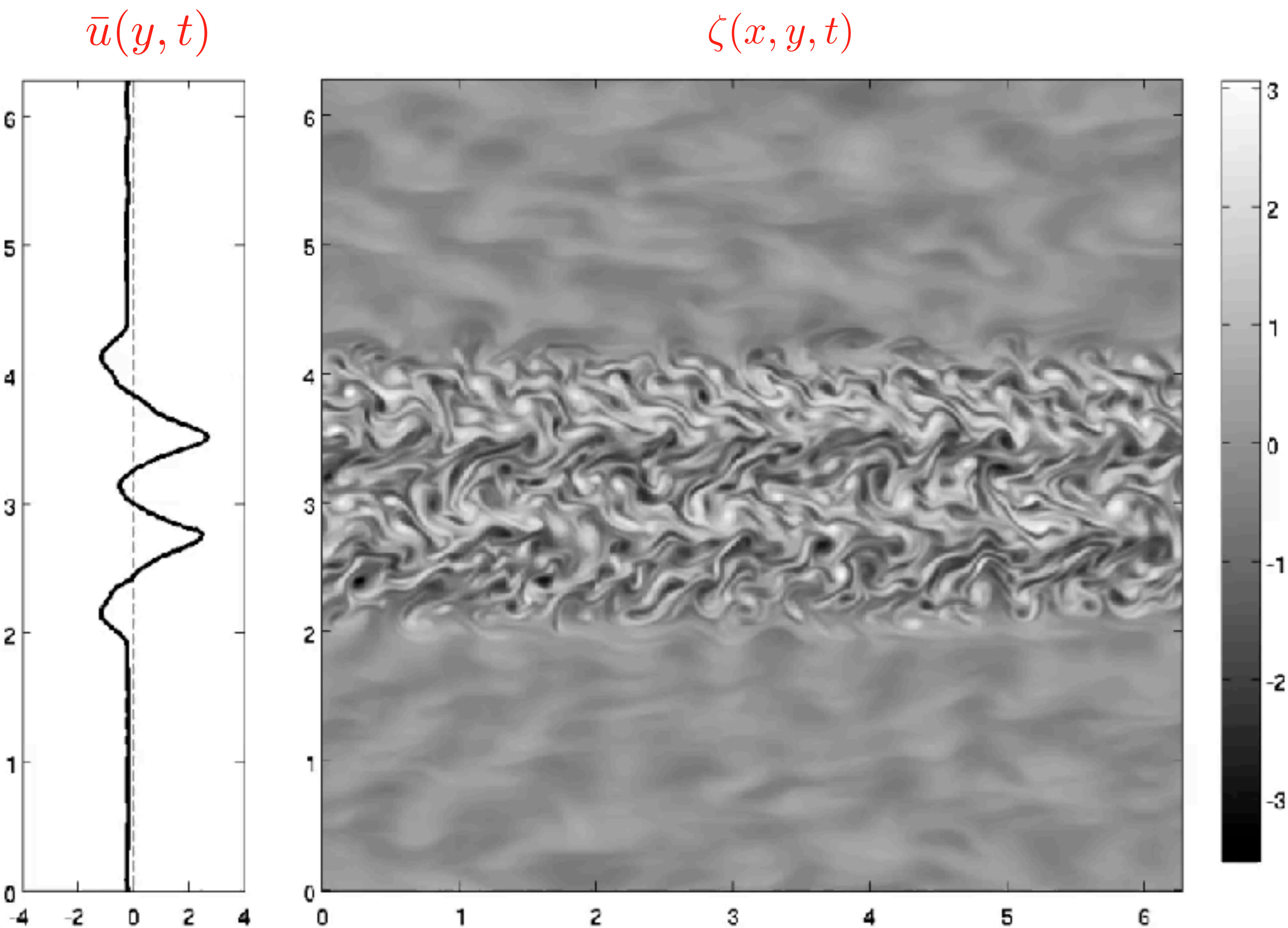
$$\bar{u} = \frac{\overline{\xi\zeta'} - \mu\overline{\zeta'^2} - \nu\overline{|\nabla\zeta'|^2}}{\mu\beta}$$

Westward flow in unforced regions

This result is correct, but it's not the whole story.

Let's increase the width of the forced strip.

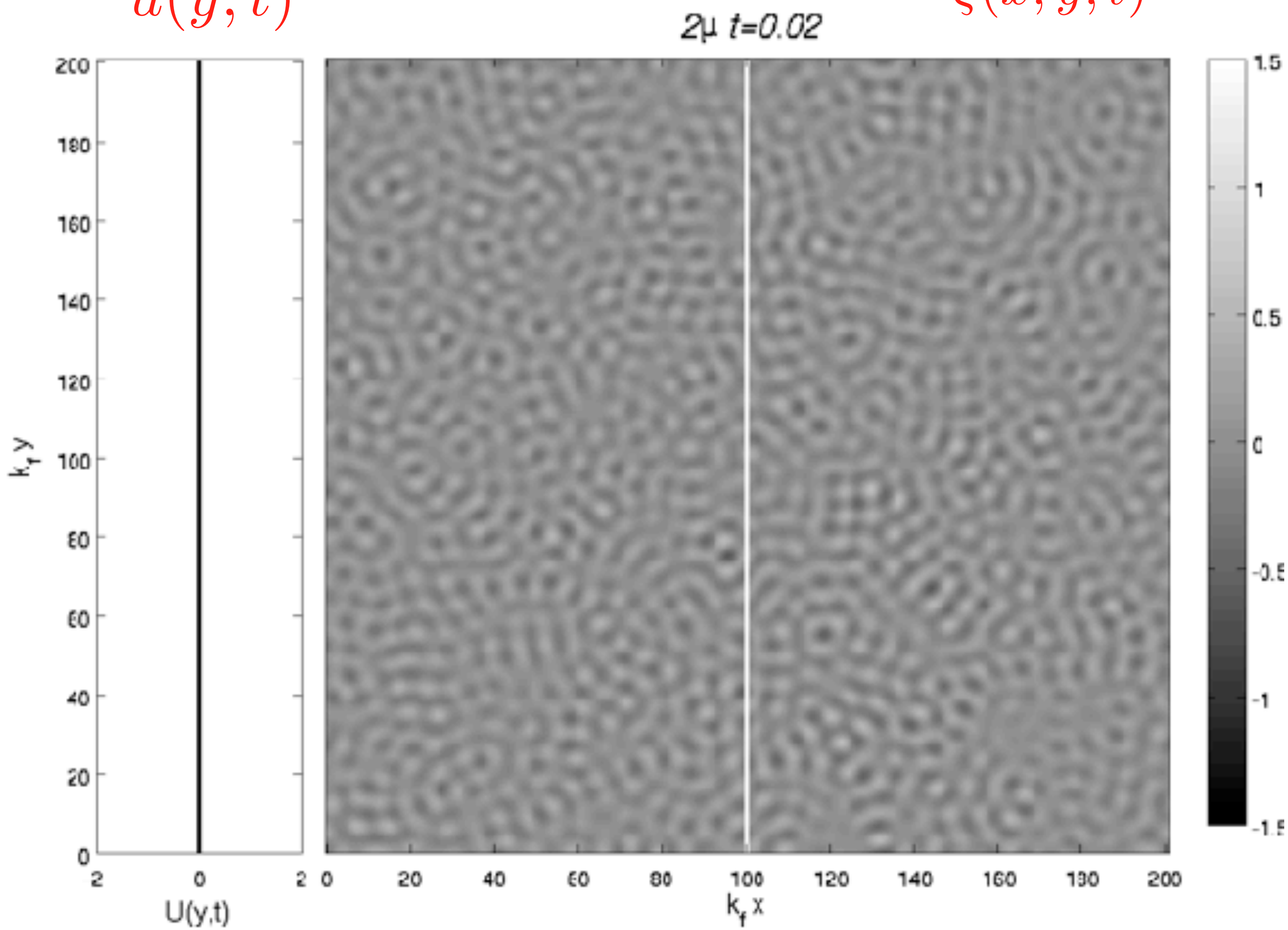




$\bar{u}(y,t)$
$\zeta(x,y,t)$

β-plane turbulence driven by white noise forcing

$\bar{u}(y,t)$ $\zeta(x,y,t)$



At the end there are seven jets: there is scale separation from the forcing.

Srinivasan ~2014, following many other authors

$\beta_* = 1$ $\mu_* = 0.0182$ $Z = \dfrac{\varepsilon \beta^2}{\mu^5} = 1.2 \times 10^6$ $k_f L = 32$

What did we just see?

There is an underlying spatially homogeneous turbulent flow. But this flow is unstable to formation of jets. The jets initially grow exponentially and then saturate at finite amplitude.

The mature jets are strong and the turbulence is no longer homogenous e.g., because of the jet shear.

This is "zonostrophic instability". It can also be viewed as "negative viscosity" or "anti-friction"

The zonal-mean momentum equation

Zonally average the QGPV eqn, and then integrate.

$\bar{\zeta} = -\bar{u}_y$

$$\bar{\zeta}_t + \left(\overline{v'\zeta'}\right)_y = -\mu\bar{\zeta} + \nu\bar{\zeta}_{yy}$$

We have made a great leap backwards to the momentum equation. Note the “eddy vortex force”.

$$\bar{u}_t = \overline{v'\zeta'} - \mu\bar{u} + \nu\bar{u}_{yy}$$

The eddy vortex force is related to the Reynolds stress.

$$\overline{v'\zeta'} = -\left(\overline{u'v'}\right)_y$$

(Taylor’s identity)

If we use an eddy viscosity closure then, “on average”, the eddy viscosity is negative.

$$\overline{u'v'} = -\nu_e\bar{u}_y$$

$$\langle\overline{u'v'}\,\bar{u}_y\rangle = \mu\langle\bar{u}^2\rangle$$

(The mean energy equation)

An important property of white-noise forcing

$$\zeta_t + u\zeta_x + v\zeta_y + \beta v = \underbrace{\xi}_{\text{forcing}} - \underbrace{\mu\zeta}_{\text{drag}} + \nu\nabla^2\zeta$$

The energy power integral is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle u^2 + v^2\rangle \underbrace{-\langle\psi\xi\rangle}_{\varepsilon} = \mu\langle u^2 + v^2\rangle + \nu\langle\zeta^2\rangle$$

White noise forcing specifies energy injection, ε. Drag is required to achieve statistical steady state.

$$U_{RMS} = \sqrt{\frac{\varepsilon}{\mu}} \text{ is known}$$

$$\therefore \; L_{\text{Rhines}} = \frac{\varepsilon^{1/4}}{\beta^{1/2}\mu^{1/4}}$$

This is the predicted jet scale.

But this Rhines length is not the only length scale.

$$L_{\mathrm{Rhines}} = \left(\frac{\varepsilon}{\beta^2 \mu}\right)^{1/4}$$

Halting the inverse cascade

Drag is scale selective — it acts heavily on the biggest, slowest eddies and halts the inverse cascade (with and without beta).

$$\ell_{\text{Lilly}} = \sqrt{\frac{\varepsilon}{\mu^3}}$$

Lilly's length scale applies to 2D turbulence with β=0.

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

$$U_k = \sqrt{\int_{k/\sqrt{2}}^{\sqrt{2}k} E(k')\mathrm{d}k'} \sim \varepsilon^{1/3}k^{-1/3}$$

$$T_k = \frac{1}{kU_k} \sim \varepsilon^{-1/3}k^{-2/3}$$

$$\mu T_k \sim 1 \quad \Rightarrow \quad k \sim \sqrt{\frac{\mu^3}{\varepsilon}}$$

Lilly 1969

β can slow down the inverse cascade, and funnel it into ZM flow. But β alone cannot halt the cascade. (I believe.)

The "zonostrophy number"

(Z is not standard terminology)

$$Z \stackrel{\mathrm{def}}{=} \frac{\varepsilon \beta^2}{\mu^5}$$

Sukoriansky, Dikovskaya & Galperin 2007

$$\left.\begin{aligned} \dim\, \varepsilon &= \tfrac{L^2}{T^3} \\ \dim\, \beta &= \tfrac{1}{LT} \\ \dim\, \mu &= \tfrac{1}{T} \end{aligned}\right\} \Rightarrow \quad \dim\left(\frac{\varepsilon\beta^2}{\mu^5}\right) = L^0 T^0$$

Three length scales:

$$\left.\begin{aligned} L_{\mathrm{VM}} &= \left(\tfrac{\varepsilon}{\beta^3}\right)^{1/5} \\ L_{\mathrm{Rhines}} &= \left(\tfrac{\varepsilon}{\beta^2 \mu}\right)^{1/4} \\ L_{\mathrm{Lilly}} &= \left(\tfrac{\varepsilon}{\mu^3}\right)^{1/2} \end{aligned}\right\} \Rightarrow \quad \frac{L_{\mathrm{Lilly}}}{L_{\mathrm{Rhines}}} = Z^{1/4} \quad \text{and} \quad \frac{L_{\mathrm{Rhines}}}{L_{\mathrm{VM}}} = Z^{1/20}$$

"L_{VM} characterizes the intensity of the forcing relative to the PV gradient."

$$L_{\mathrm{Lilly}} L_{\mathrm{VM}}^5 = L_{\mathrm{Rhines}}^6$$

What can we say about the structure of the zonal jets?

Motivated by Gas-Giant atmospheres, there has been interest in the limiting case

$$Z \stackrel{\text{def}}{=} \frac{\varepsilon\beta^2}{\mu^5} \gg 1$$

This case has not much to do with the atmosphere and ocean. But I'll briefly discuss it. Warning: this is speculative

PV staircases

$$Z \stackrel{\text{def}}{=} \frac{\varepsilon \beta^2}{\mu^5} \sim 10^{20} \text{ !?}$$

Marcus (1993)

Strong eddies mix PV into homogeneous layers.

The PV jumps are mixing barriers.

Uniform PV produces a parabolic velocity profile:

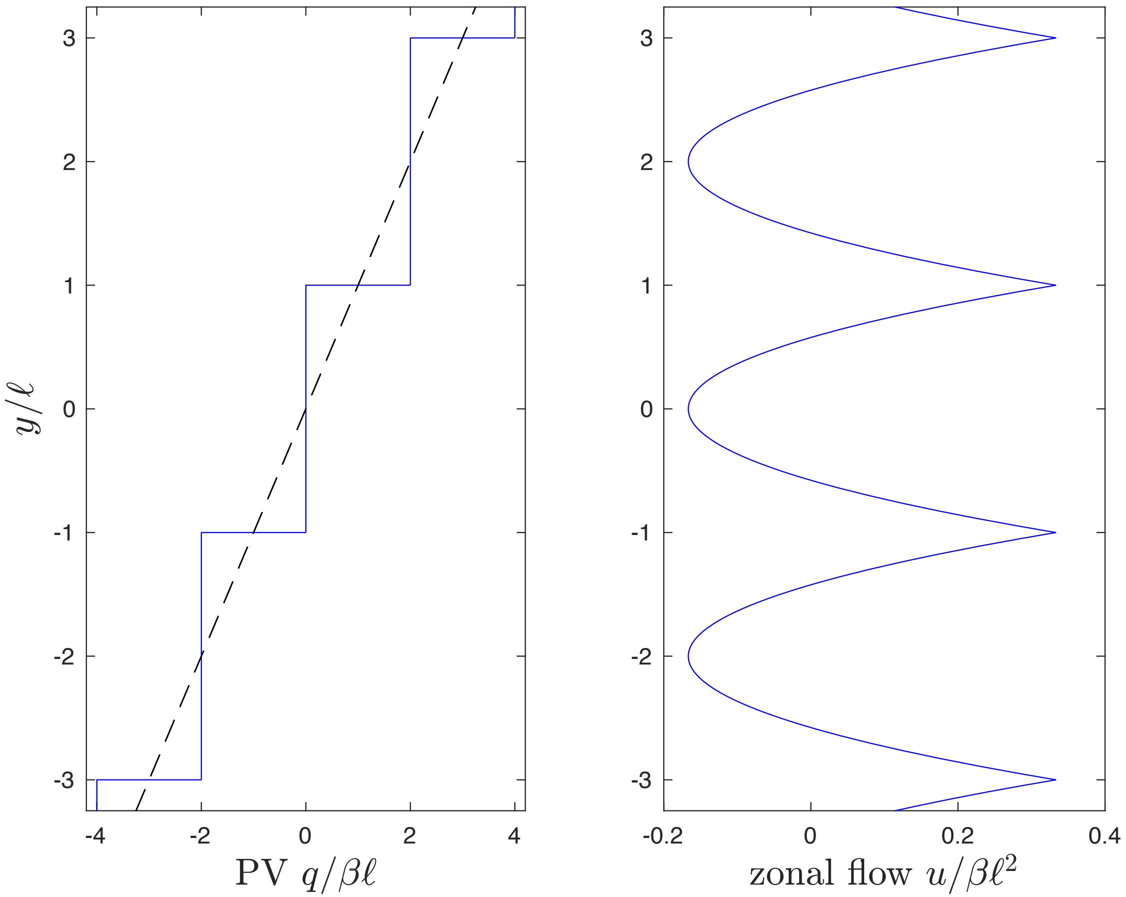
$$\bar{u} = \tfrac{1}{2}\beta y^2 - \tfrac{1}{6}\beta \ell^2 \,,$$

$$\text{for} \quad -\ell < y < \ell$$

Note $\displaystyle\int_{-\ell}^{\ell} \bar{u}\, \mathrm{d}y = 0$

The step thickness is determined from the energy power integral

$$\langle \bar{u}^2 \rangle = \frac{\beta^2 \ell^4}{45} \approx \frac{\varepsilon}{\mu}$$



A useful rule of thumb:

$$u_{\text{east}} = -2u_{\text{west}} = \tfrac{1}{3}\beta \ell^2$$

A PV staircase?

$$Z \stackrel{\text{def}}{=} \frac{\varepsilon \beta^2}{\mu^5}$$

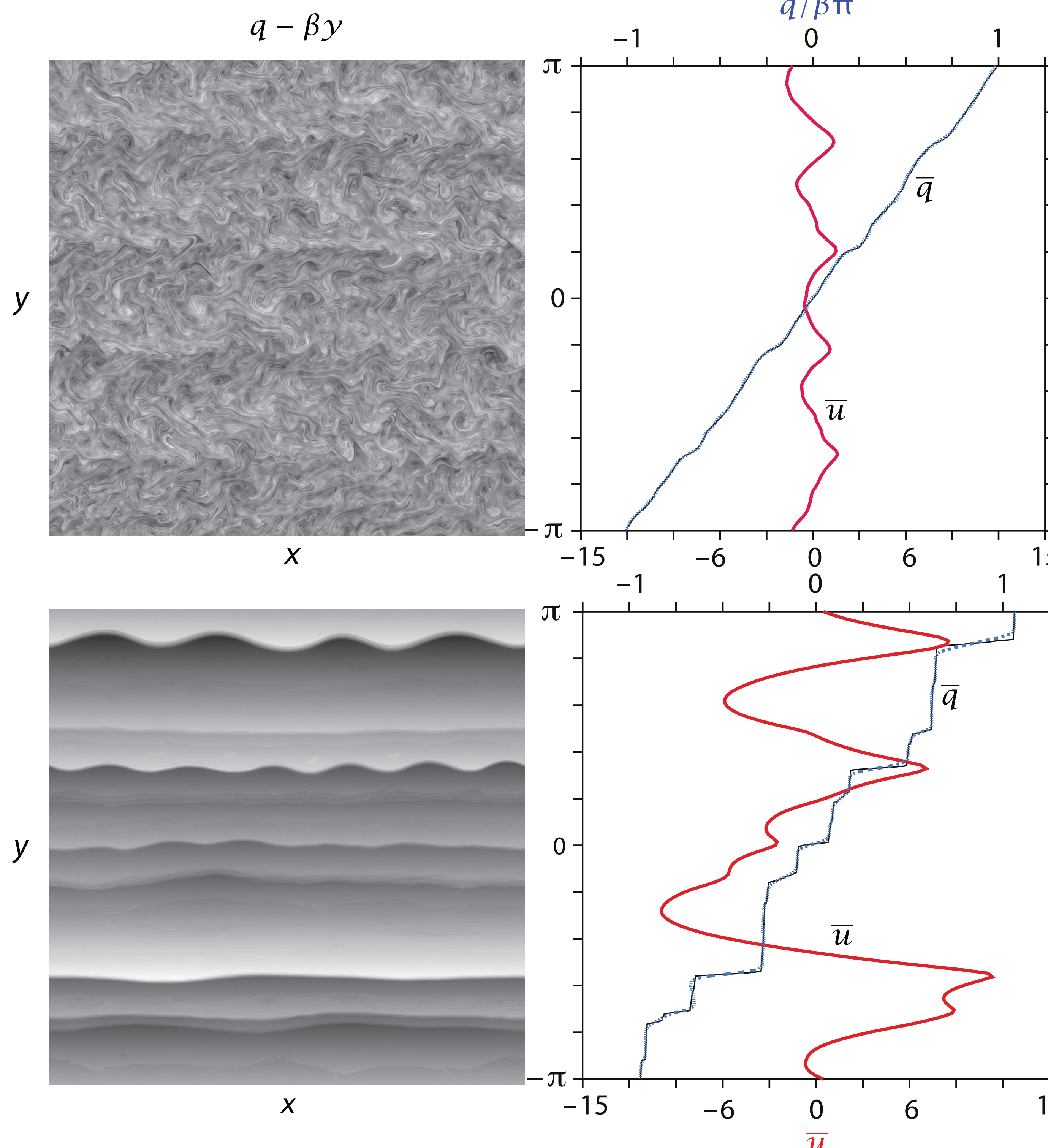
$$= \left(\frac{L_{\text{Rhines}}}{L_{\text{VM}}} \right)^{20}$$

$$= 3^{20} = 3.5 \times 10^9$$

Forcing is via random injection of vortex dipoles and is broad-band in physical space.

$$Z = 11^{20} = 6.7 \times 10^{20}$$

But this staircase does not meet the building code.



Adapted from Scott & Dritschel (2012) by Vallis (2017)

Note $L_{\text{Lilly}} = Z^{1/4} L_{\text{Rhines}} \gg L_{\text{Rhines}}$

End of forced barotropic β-plane turbulence

It is very easy to produce zonal jets
— any sort of forcing will do it.

But we there is not a good understanding of how the jets and eddies depend on the non-dimensional parameter $Z=\varepsilon\beta^2/\mu^5$.

Is it true that the wavenumber of the forcing is “irrelevant”?

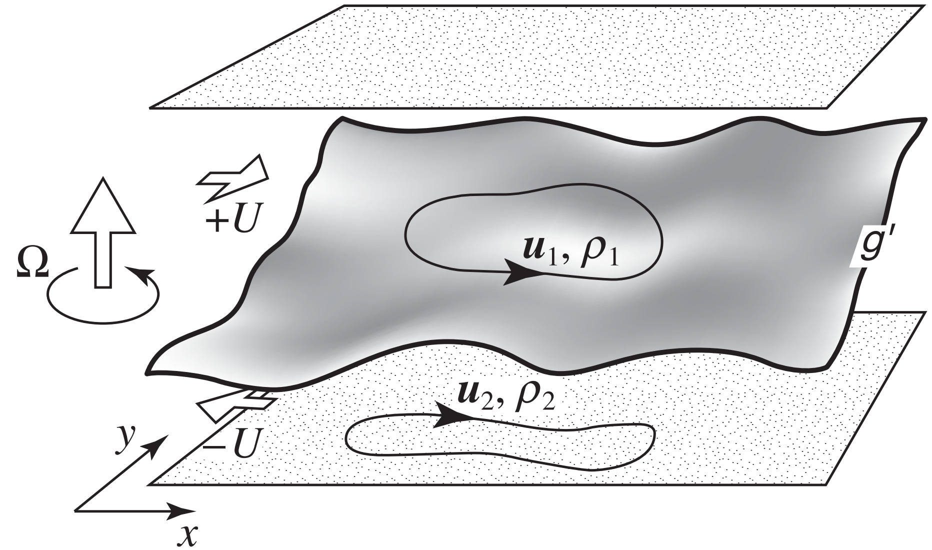
We can't say much about this problem so let's move on to a more difficult one…

THE END?

Baroclinic turbulence

(Some background for Isaac Held's lectures next week.)

The two-layer QG model



"Reduced gravity" $g' = \frac{\rho_2 - \rho_1}{\rho_1} g$

Conservation of PV in each layer

$$\partial_t q_1 + \psi_{1x} q_{1y} - \psi_{1y} q_{1x} = \kappa \triangle q_1$$

$$\partial_t q_2 + \psi_{2x} q_{2y} - \psi_{2y} q_{2x} = \kappa \triangle q_2 - \mu \nabla^2 \psi_2$$

$$q_1 = \triangle \psi_1 + \alpha_2 \, k_\mathrm{d}^2 (\psi_2 - \psi_1) + \beta y$$

$$q_2 = \triangle \psi_1 + \alpha_1 \, k_\mathrm{d}^2 (\psi_1 - \psi_2) + \beta y$$

The "Rossby deformation wavenumber"

$$k_\mathrm{d}^2 = \frac{f_0 (H_1 + H_2)}{g' H_1 H_2}$$

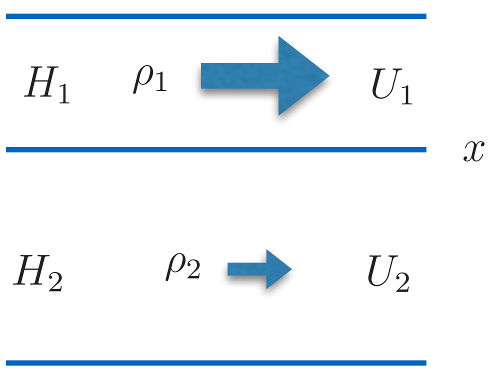
Layer thickness ratios — usually taken as 1/2

$$(\alpha_1, \alpha_2) = \frac{(H_1, H_2)}{H_1 + H_2}$$

The "standard model" of baroclinic turbulence

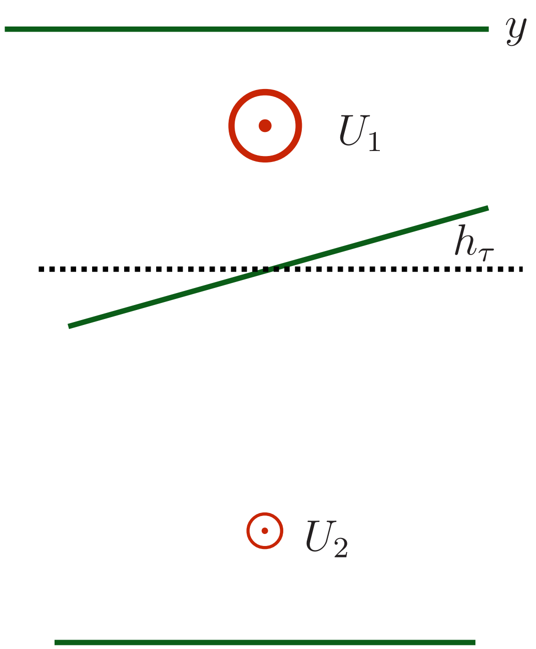
The base state is a vertically sheared zonal flow

$$\psi_n \mapsto -U_n y + \psi_n$$



Through thermal-wind balance, the "thermocline" is tilted. The base state has Available Potential Energy.

$$h_\tau = \underbrace{\frac{f_0}{g'}(U_1 - U_2)}_{\text{thermocline slope}}\; y$$



The linear theory

$$\psi_n \mapsto -U_n y + \psi_n$$

The PVs are

$$q_1 \mapsto \underbrace{\left[\beta + \alpha_2 k_{\mathrm{d}}^2 (U_1 - U_2)\right]}_{\beta_1} y + \underbrace{\triangle\psi_1 + \alpha_2 k_{\mathrm{d}}^2 (\psi_2 - \psi_1)}_{q_1}$$

$$q_2 \mapsto \underbrace{\left[\beta + \alpha_1 k_{\mathrm{d}}^2 (U_2 - U_1)\right]}_{\beta_2} y + \underbrace{\triangle\psi_2 + \alpha_2 k_{\mathrm{d}}^2 (\psi_1 - \psi_2)}_{q_1}$$

The linearized equations are

$$q_{1t} + U_1 q_{1x} + \beta_1 v_1 = 0$$
$$q_{2t} + U_2 q_{2x} + \beta_2 v_2 = 0$$

The usual approach

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly}$$

This produces an eigenproblem that can be solved exactly. Instability requires $\beta_1\beta_2<0$.
Let's consider an supplementary approach.

“Enstrophy power integrals”

Define a domain average

$$\langle \bullet \rangle = \text{average over the domain}$$

$$\langle \partial_x \bullet \rangle = \langle \partial_y \bullet \rangle = 0$$

The PV fluxes satisfy an identity.
Show this is “physically obvious”.

$$(\alpha_1, \alpha_2) = \frac{(H_1, H_2)}{H_1 + H_2}$$

$$\langle v_1 q_1 \rangle = \alpha_2 k_{\mathrm{d}}^2 \langle \psi_{1x}(\psi_2 - \psi_1) \rangle$$

$$\langle v_2 q_2 \rangle = \alpha_1 k_{\mathrm{d}}^2 \langle \psi_{2x}(\psi_1 - \psi_2) \rangle$$

$$\Rightarrow \quad \alpha_1 \langle v_1 q_1 \rangle + \alpha_2 \langle v_2 q_2 \rangle = 0$$

From the linear equations
we can easily show that

$$\partial_t \langle \tfrac{1}{2} q_1^2 \rangle + \beta_1 \langle v_1 q_1 \rangle = 0$$

$$\partial_t \langle \tfrac{1}{2} q_2^2 \rangle + \beta_2 \langle v_2 q_2 \rangle = 0$$

Use the red identity to
get a conservation law.
(pseudomomentum?)

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\alpha_1}{\beta_1} \langle \tfrac{1}{2} q_1^2 \rangle + \frac{\alpha_2}{\beta_2} \langle \tfrac{1}{2} q_2^2 \rangle \right] = 0$$

The enstrophy conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\alpha_1}{\beta_1}\left\langle\tfrac{1}{2}q_1^2\right\rangle + \frac{\alpha_2}{\beta_2}\left\langle\tfrac{1}{2}q_2^2\right\rangle\right] = 0$$

For an exponentially growing normal mode the red identity implies

$$\frac{\alpha_1}{\beta_1}\left\langle\tfrac{1}{2}q_1^2\right\rangle + \frac{\alpha_2}{\beta_2}\left\langle\tfrac{1}{2}q_2^2\right\rangle = 0$$

A necessary condition for normal mode instability is that the PV gradients have opposite signs.

$$\therefore\ \beta_1\beta_2 < 0$$

Limit attention to eastward flow in the top layer

$$\beta_1 = \beta + \alpha_2 k_\mathrm{d}^2 (U_1 - U_2) > 0$$
$$\beta_2 = \beta + \alpha_1 k_\mathrm{d}^2 (U_1 - U_2) < 0$$

“Non-linearize” around the base state

$$\psi_n \mapsto -U_n y + \psi_n$$

The PVs are

$$q_1 \mapsto \underbrace{\left[\beta + \alpha_2 k_{\mathrm{d}}^2 (U_1 - U_2)\right]}_{\beta_1} y + \underbrace{\triangle\psi_1 + \alpha_2 k_{\mathrm{d}}^2 (\psi_2 - \psi_1)}_{q_1}$$

$$q_2 \mapsto \underbrace{\left[\beta + \alpha_1 k_{\mathrm{d}}^2 (U_2 - U_1)\right]}_{\beta_2} y + \underbrace{\triangle\psi_2 + \alpha_2 k_{\mathrm{d}}^2 (\psi_1 - \psi_2)}_{q_1}$$

The QGPV equation is

$$q_{1t} + U_1 q_{1x} + \beta_1 v_1 + J(\psi_1, q_1) = \kappa \triangle q_1$$

$$q_{2t} + U_2 q_{2x} + \beta_2 v_2 + J(\psi_2, q_2) = \kappa \triangle q_2 - \mu \triangle \psi_2$$

This is a popular model of homogenous baroclinic turbulence.

The turbulence is spatially homogeneous even though the base state depends on *y*.

Discussion of this baroclinic turbulence model

(Rhines, Salmon, Held, Larichev, Lapeyre, Thompson, Young)

The good

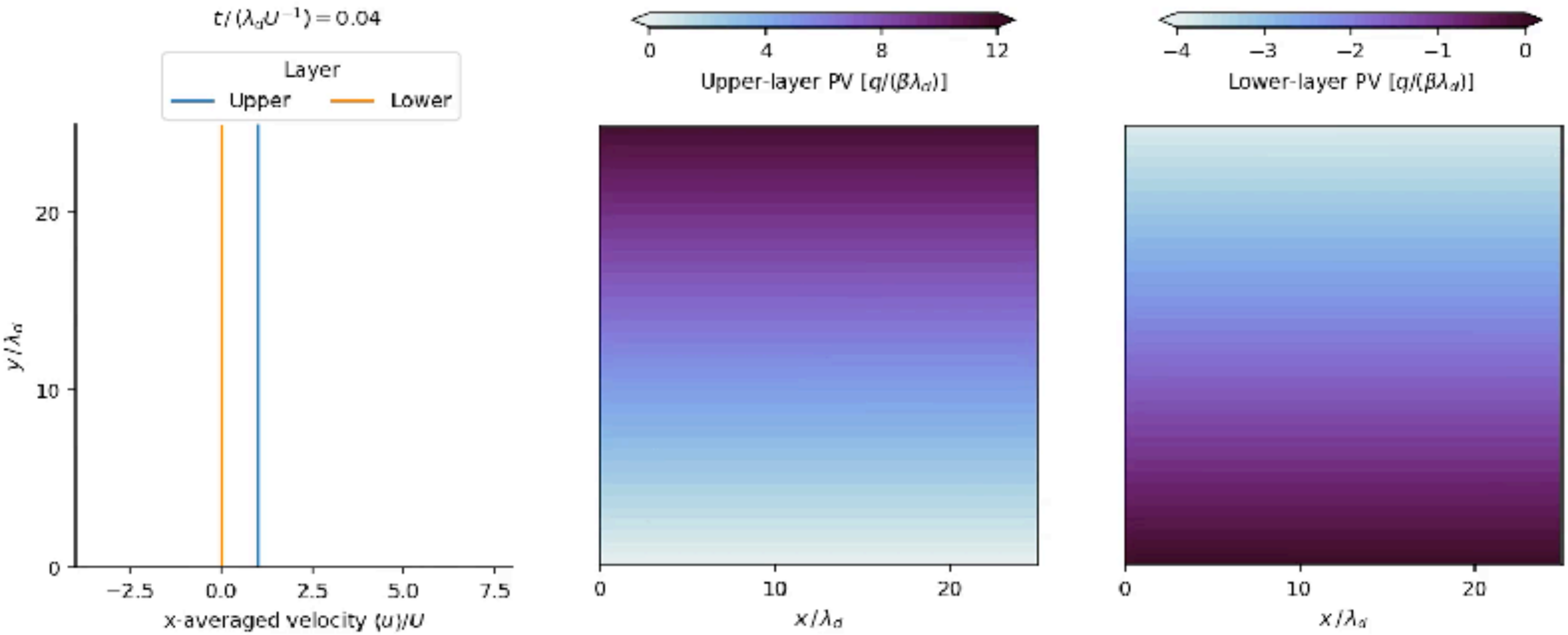
The forcing, U_1 - U_2, is more physically realistic than the white-noise agitation used in the barotropic model. Note that the energy input is not specified in advance.

The bad

The turbulence cannot equilibrate by reducing the vertical shear: U_1 and U_2 are held fixed. OTOH, the turbulence does equilibrate!

The ugly

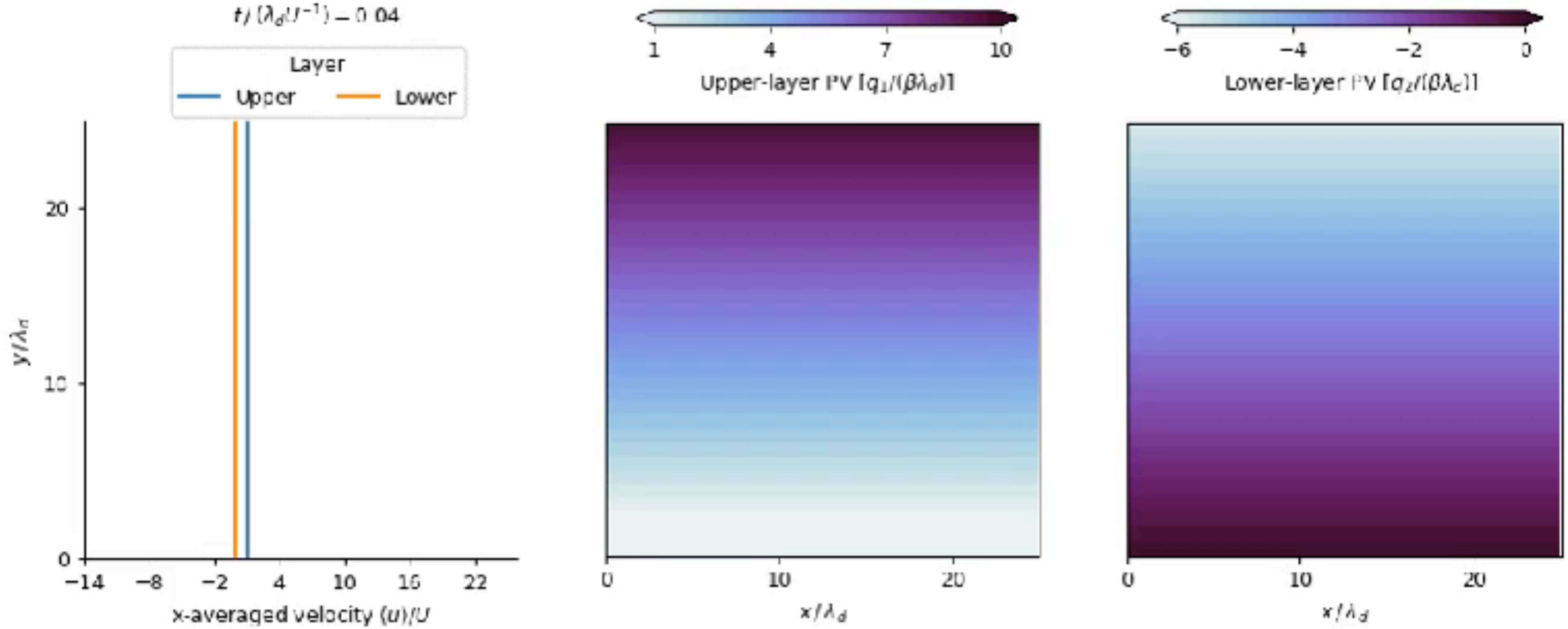
The equilibration mechanism is not clear. And we can't answer basic questions e.g., how does the energy input depend on U_1 - U_2 ?



$$\tfrac{1}{2}k_{\mathrm{d}}^2 U_1 = 2\beta\,, \qquad \frac{\mu}{U_1 k_{\mathrm{d}}} = 0.04\,, \qquad 2\pi L = 25\lambda_{\mathrm{d}}\,, \qquad H_1 = H_2$$

$$\beta_1 = \beta + \tfrac{1}{2}k_{\mathrm{d}}^2 U_1\,, \\ = 3\beta$$

$$\beta_2 = \beta - \tfrac{1}{2}k_{\mathrm{d}}^2 U_1\,, \\ = -\beta$$



$$\tfrac{1}{2}k_{\mathrm{d}}^2 U_1 = 4\beta\,, \qquad \frac{\mu}{U_1 k_{\mathrm{d}}} = 0.04\,, \qquad 2\pi L = 25\lambda_{\mathrm{d}}\,, \qquad H_1 = H_2$$

$$\beta_1 = \beta + \tfrac{1}{2}k_{\mathrm{d}}^2 U_1\,, \quad = 5\beta \qquad\qquad \beta_2 = \beta - \tfrac{1}{2}k_{\mathrm{d}}^2 U_1\,, \quad = -3\beta$$

PV flux identities

Start with the "inversion relation"

$$\tau \stackrel{\text{def}}{=} \psi_1 - \psi_2$$

$$q_1 = \zeta_1 - \alpha_2 k_\mathrm{d}^2 \tau \,,$$
$$q_2 = \zeta_2 + \alpha_1 k_\mathrm{d}^2 \tau \,.$$

With I by P obtain "heat flux" identities

$$v \stackrel{\text{def}}{=} \alpha_1 v_1 + \alpha_2 v_2$$

$$\langle v_1 \tau \rangle = \langle v_2 \tau \rangle = \langle v \tau \rangle$$

PV fluxes can be written in terms of the heat flux

$$\langle v_1 q_1 \rangle = -\alpha_2 k_\mathrm{d}^2 \langle v \tau \rangle$$
$$\langle v_2 q_2 \rangle = +\alpha_1 k_\mathrm{d}^2 \langle v \tau \rangle$$

$$\mathcal{F} \stackrel{\text{def}}{=} \alpha_1 \alpha_2 k_\mathrm{d}^2 \langle v \tau \rangle$$
$$= -\alpha_1 \langle v_1 q_1 \rangle$$
$$= +\alpha_2 \langle v_2 q_2 \rangle$$

Now use the QGPV equations

$$q_{1t} + U_1 q_{1x} + \beta_1 v_1 + J(\psi_1, q_1) = \kappa \triangle q_1$$

$$q_{2t} + U_2 q_{2x} + \beta_2 v_2 + J(\psi_2, q_2) = \kappa \triangle q_2 - \mu \triangle \psi_2$$

Two enstrophy power integrals (assume $\beta_1 > 0$)

$$\mathcal{F} = -\alpha_1 \langle v_1 q_1 \rangle = \alpha_2 \langle v_2 q_2 \rangle$$

$$\beta_1 \mathcal{F} = \alpha_1 \kappa \langle |\nabla q_1|^2 \rangle > 0$$

$$\beta_2 \mathcal{F} = -\alpha_2 \kappa \langle |\nabla q_2|^2 \rangle - \alpha_2 \mu \langle \zeta_2 q_2 \rangle$$

The energy power integral

$$\varepsilon = (U_1 - U_2)\mathcal{F}$$

$$(U_1 - U_2)\mathcal{F} = \alpha_2 \mu \langle \zeta_2^2 \rangle$$

$$+\kappa \langle \alpha_1 \zeta_1^2 + \alpha_2 \zeta_2^2 + \alpha_2 \alpha_1$$

The heat flux $\mathcal{F}$ is the most important summary statistic.

Next week Isaac Held will discuss scaling laws for the heat flux.

$$\mathcal{F} \propto \beta_1^{?}$$