The fluctuation-dissipation theorem:
from statistical physics to climate dynamics?

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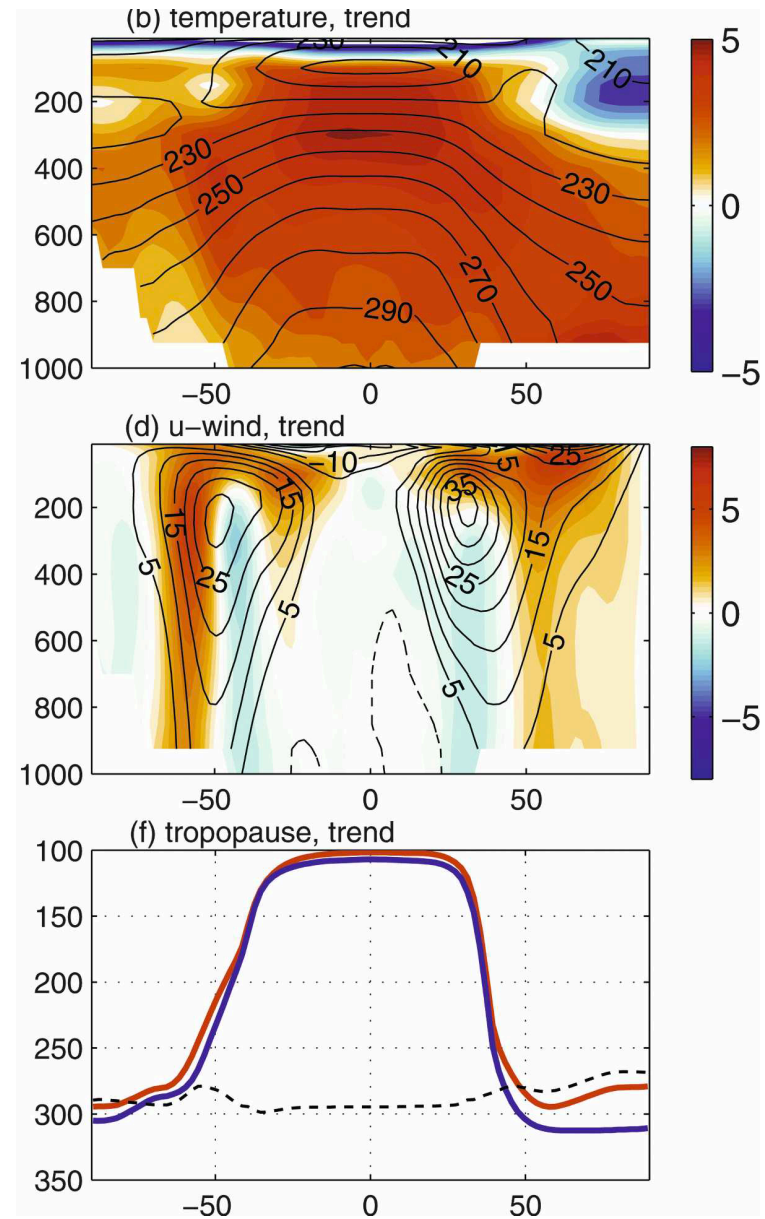
(collaborations with Fenwick Cooper and Peter Hitchcock)

The climate response problem

Lu et al 2008: simulations with GFDL CM2.1 model, 2081-2100 compared with 2001-2020 in A2 scenario.

(b) temperature, trend

(d) u–wind, trend

(f) tropopause, trend

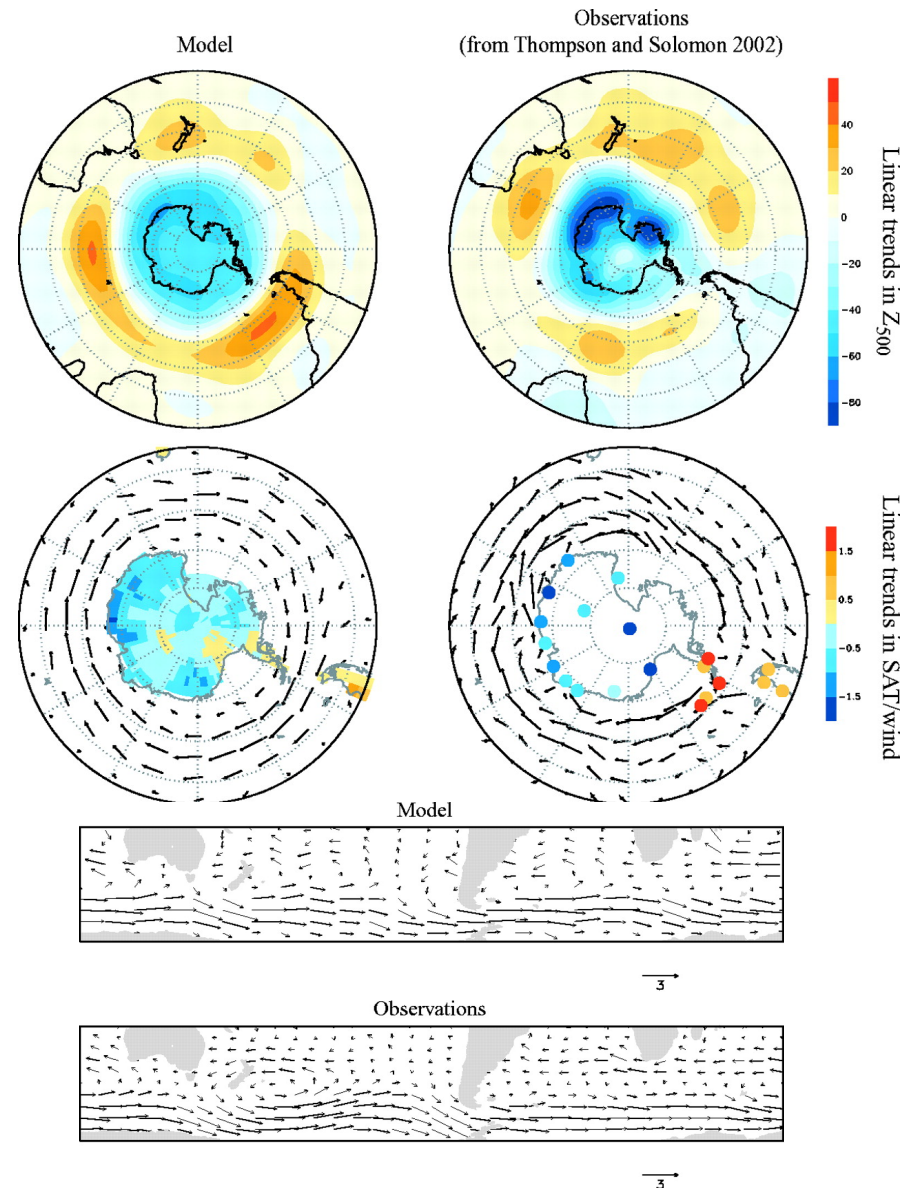Longitudinal/time average change depends on physical processes which fluctuate in longitude and time.

Changes in the SH troposphere as a dynamical response to stratospheric ozone depletion


Prescribed ozone losses (%)
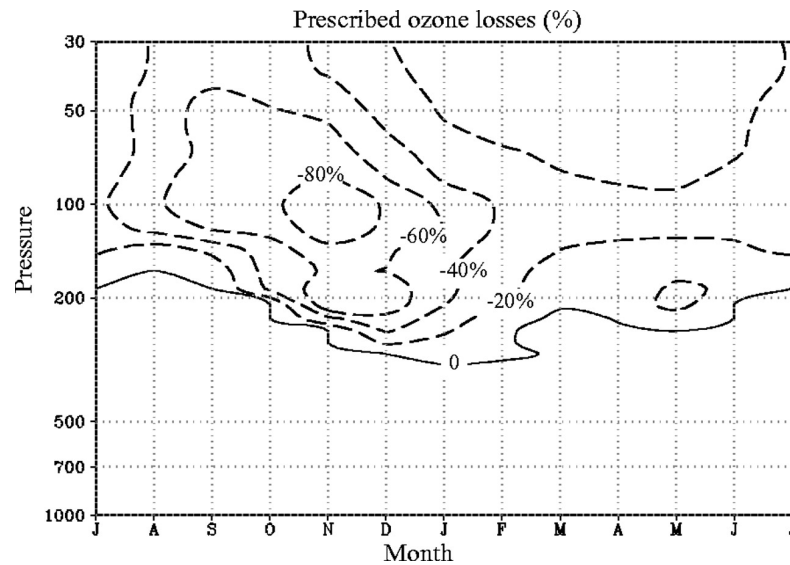
Ozone perturbation applied to AGCM

(Thompson and Solomon 2002, Gillett and Thompson, 2003)

Model

Observations
(from Thompson and Solomon 2002)

Linear trends in Z_{500}

Linear trends in SAT/wind

Model

Observations

Simple model of solar cycle effects

Haigh et al (2005): response of simple troposphere to imposed changes, e.g. uniform increase in radiative equilibrium temperature in stratosphere
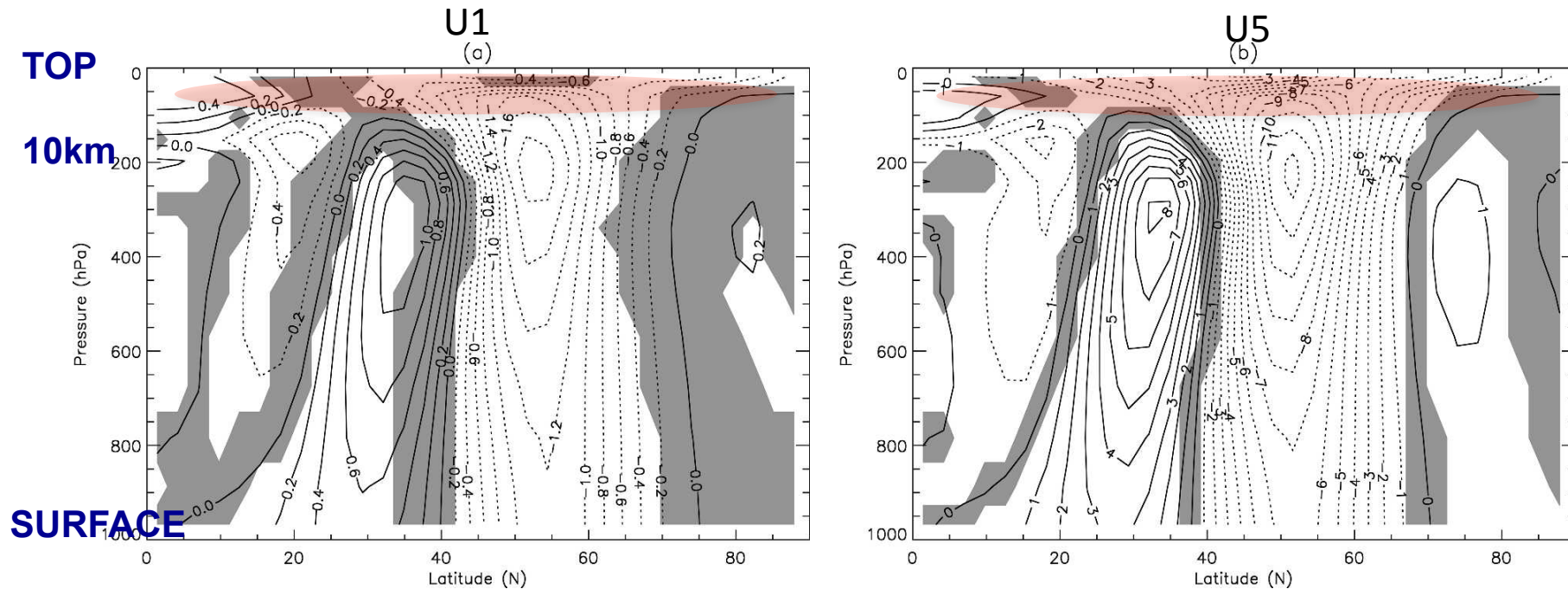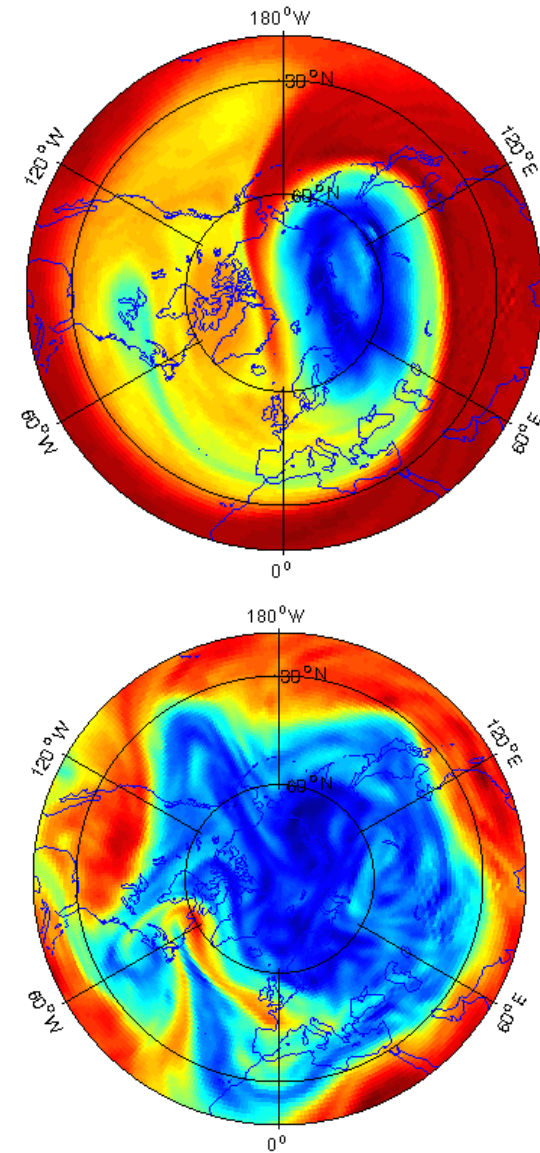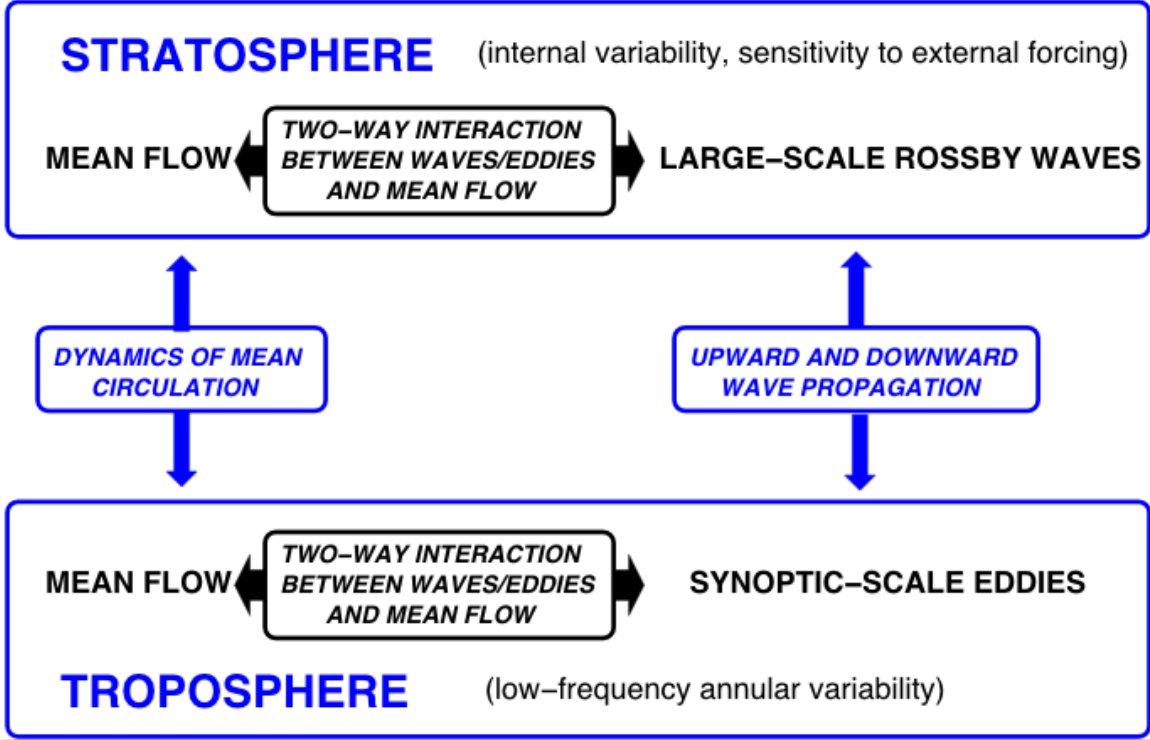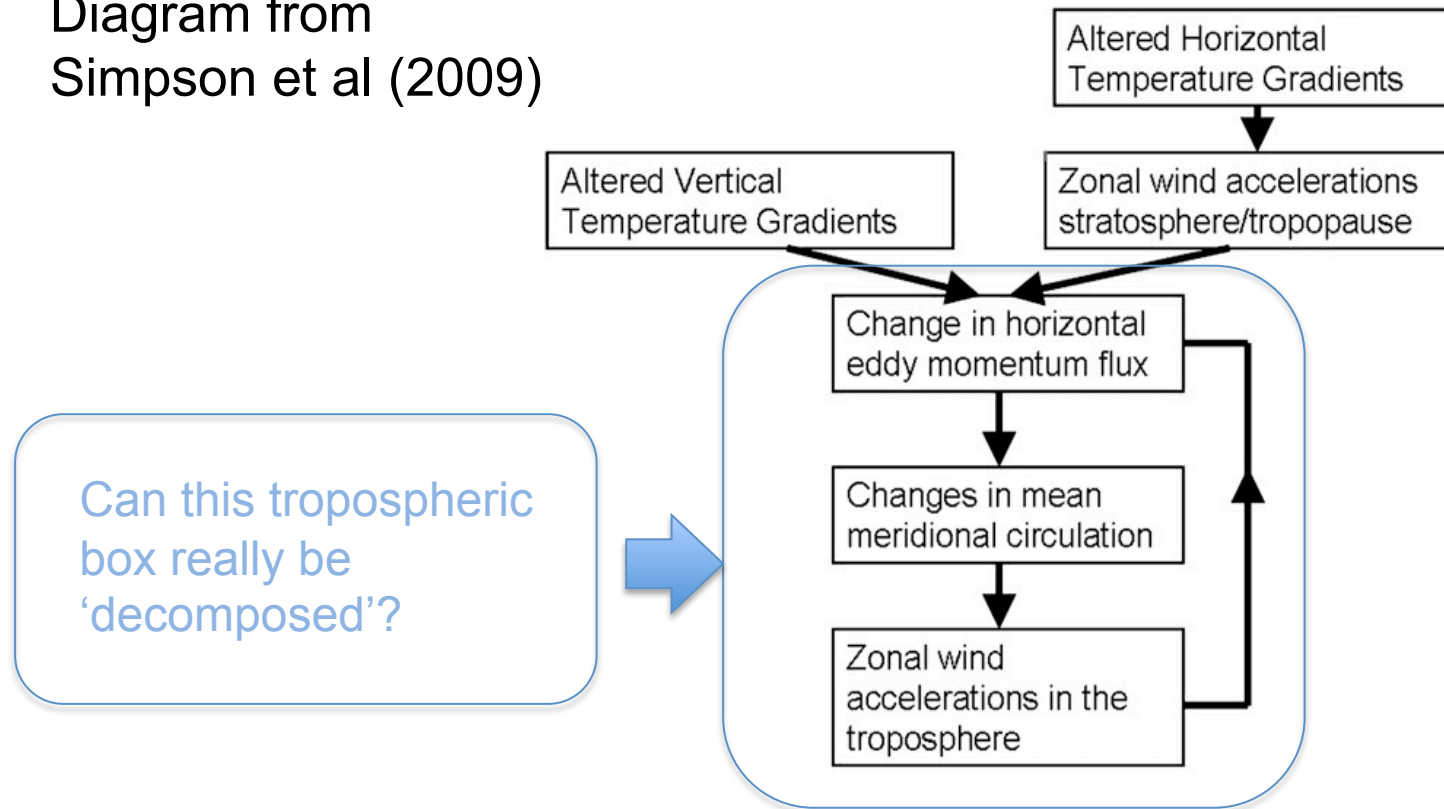
Simpson et al (2009)

Changes in eddy fluxes are a vital part of the tropospheric dynamical response (possibility of 'amplification')

U, U5, DAYS 40 TO 49 – CONTROL

E–P FLUX, U5, DAYS 40 TO 49 – CONTROL
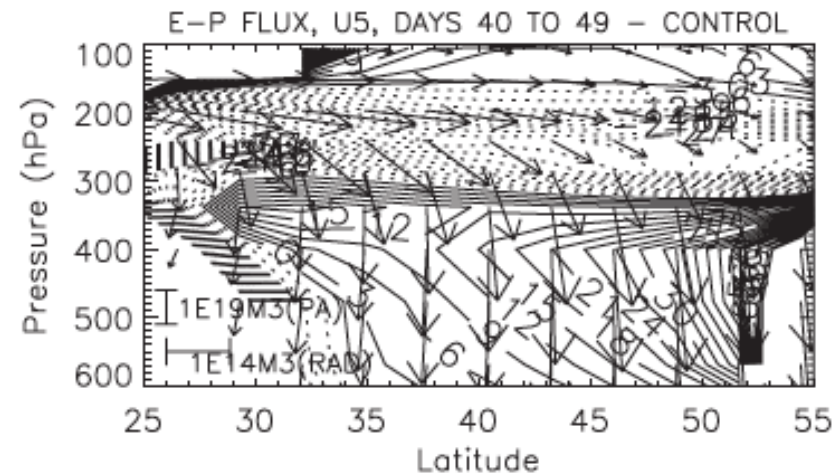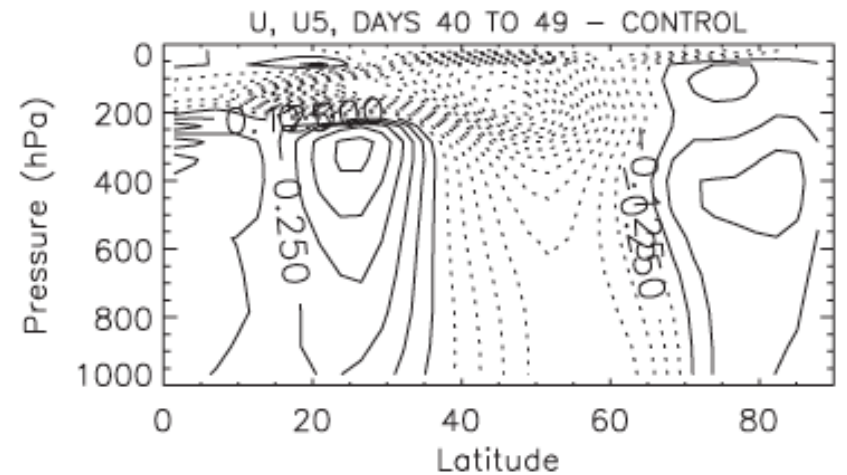
Can we make *predictions* about the response of the tropospheric circulation?

Diagram from
Simpson et al (2009)

Altered Horizontal
Temperature Gradients

Altered Vertical
Temperature Gradients

Zonal wind accelerations
stratosphere/tropopause

Change in horizontal
eddy momentum flux

Changes in mean
meridional circulation

Zonal wind
accelerations in the
troposphere

Can this tropospheric
box really be
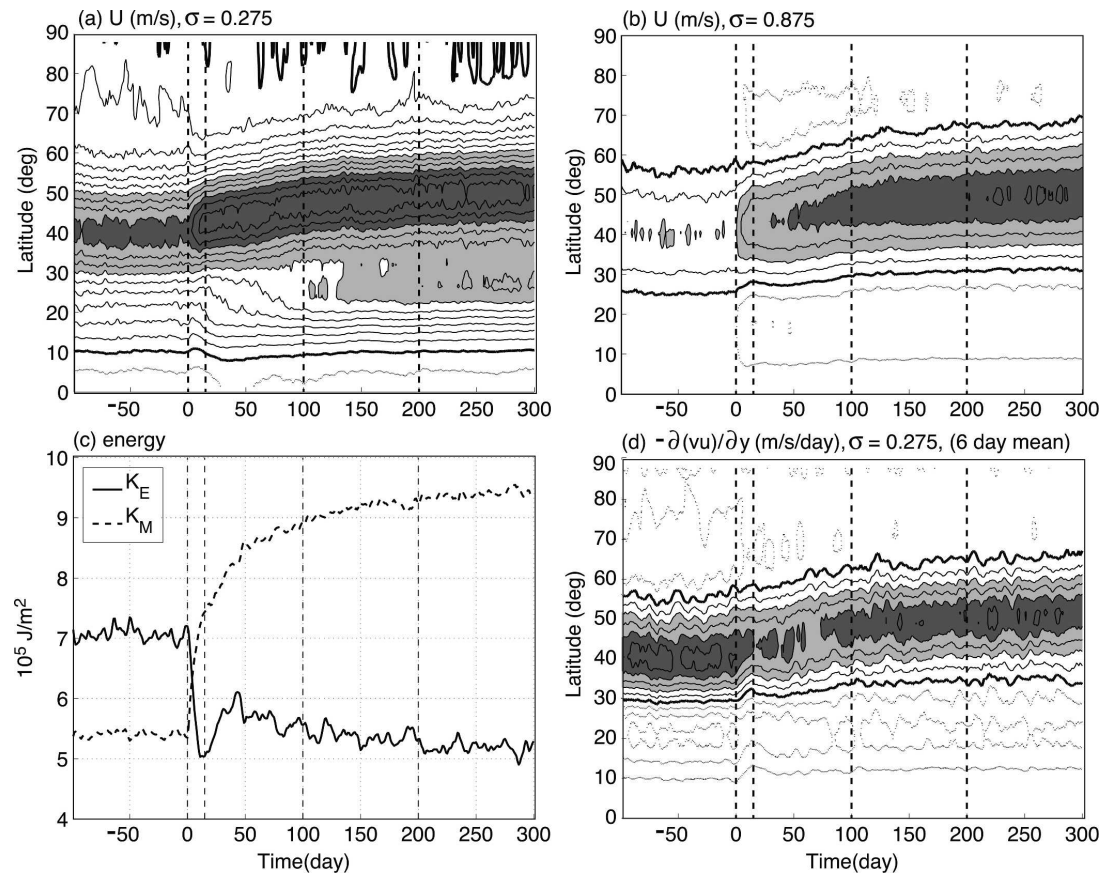'decomposed'?

Analogous tropospheric response problems: ozone hole (Gillett and Thompson 2003), stratospheric perturbation (Polvani and Kushner 2002, Song and Robinson 2004), surface friction (Chen et al 2007), tropospheric heating (Butler et al 2010)

# Reduction in surface friction in simple circulation model

Chen et al (2007): two-stage adjustment (short-time in jet strength followed by longer term change in jet position)

Chen et al (2007)

1) As the surface drag is reduced, the zonal wind acceleration is barotropic and proportional to the surface wind in the extratropics. Meanwhile, the baroclinic eddies are weakened by the increased barotropic meridional shear. But neither the weakening eddies nor the increased meridional shears are directly implicated in the poleward shift.

2) The increase in the strength of the westerlies in the extratropics leads to faster eddy phase speeds, while the subtropical zonal winds barely change. Hence, the critical latitude for these eddies is displaced poleward.

3) The dynamics of the wave breaking in the upper troposphere, in the presence of this poleward shift in critical latitude, shifts the eddy momentum fluxes poleward, driving a poleward shift in the surface zonal winds and the eddy-driven jet. This is particularly supported by the shallow water model results.

4) Eddy heat fluxes, and the associated upward Eliassen–Palm (EP) fluxes tend to follow this upper-level eddy activity. This shift in the baroclinic eddy production provides some positive feedback on the upper-level shift.
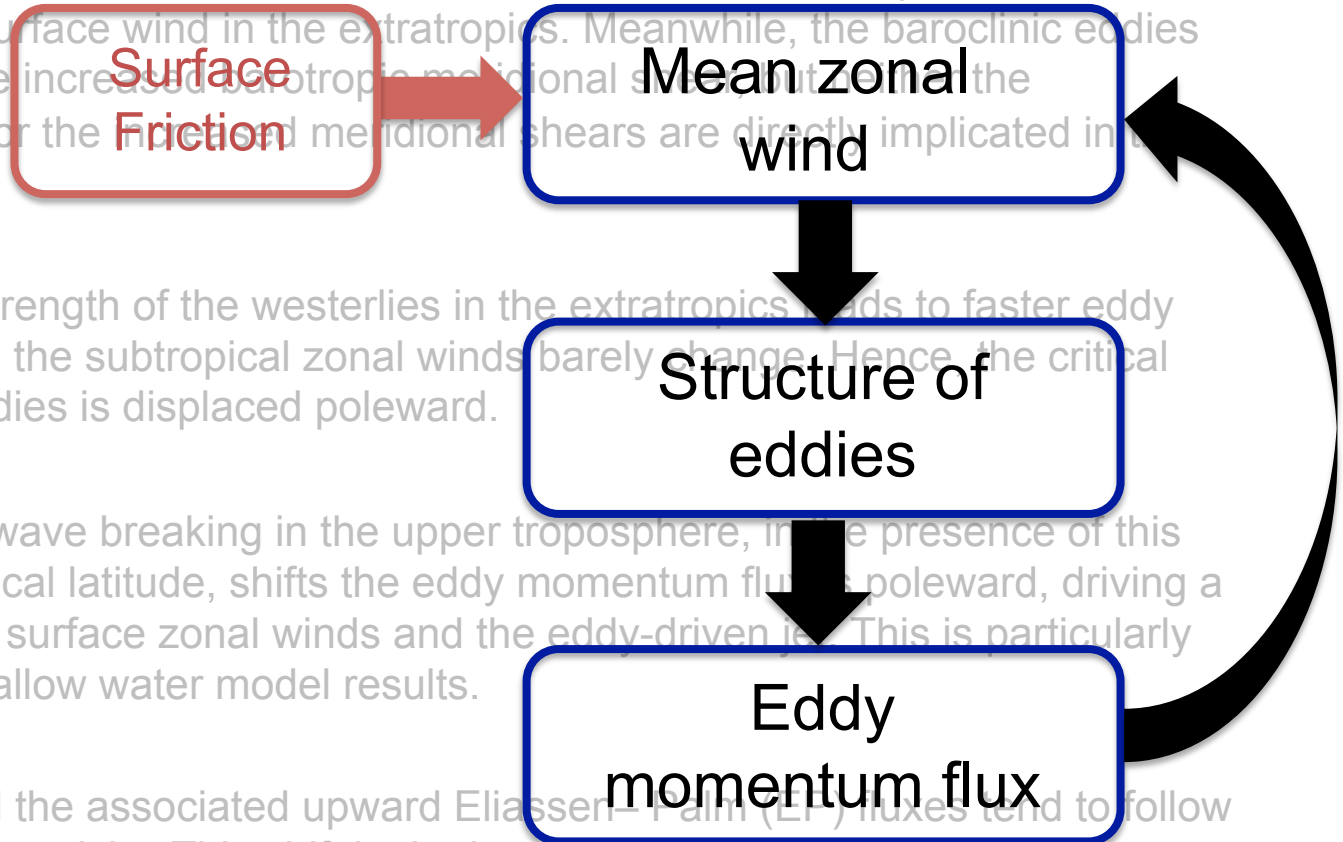
Questions

- What is relation between spatial pattern of forcing and the amplitude and spatial pattern of response? ('preferred response', 'most effective forcing')

- Will different models overpredict or underpredict response (and correct pattern of response) relative to real atmosphere?

Seek a 'unified' approach to quantitative prediction of tropospheric response, rather than post-hoc explanation of each special case
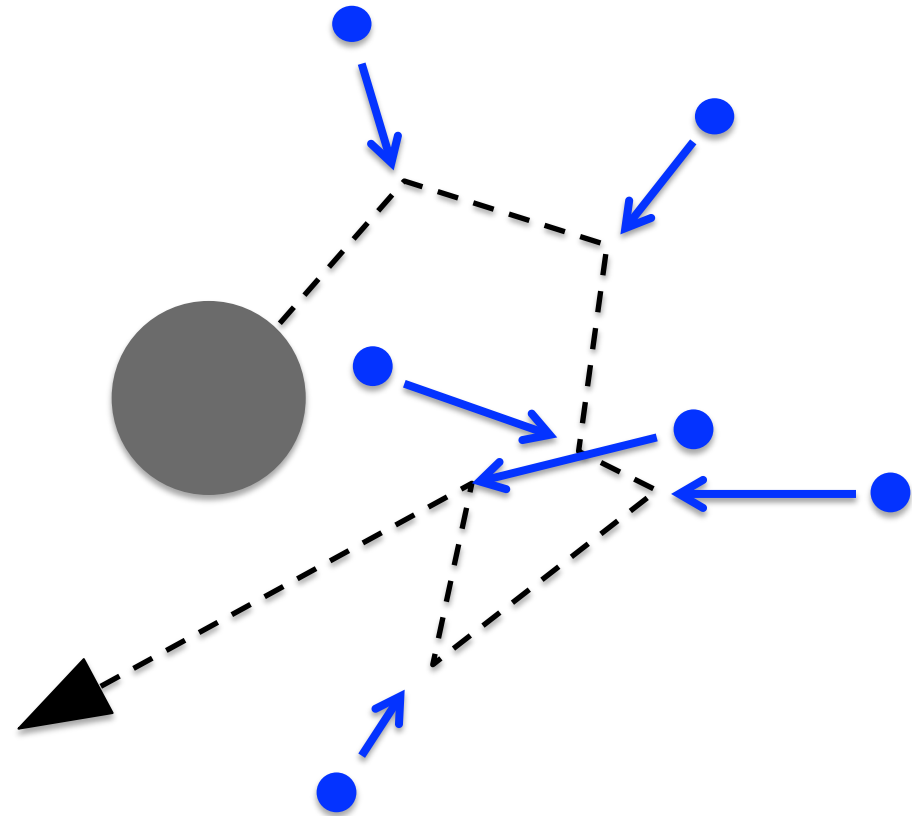
Brownian motion

Einstein (1905,1906)
Smoluchowksi (1906)

'Observable' Diffusivity of Particle:

$$D_x = \langle V_x^2 \rangle \tau_{corr}$$

Equipartition of kinetic energy: $\frac{1}{2} m \langle V_x^2 \rangle = \frac{1}{2} \frac{RT}{N_A}$

Stokes law for viscous drag force: $F_S = -k_S V = -6\pi\mu a V$

$$m\frac{dV_x}{dt} = -k_S V_x + f_R(t)$$

$$\langle f_R(t_1)f_R(t_2)\rangle = C\delta(t_1 - t_2)$$

$$\langle V_x(t_1)V_x(t_2)\rangle = \langle V_x^2\rangle \exp(-k_S|t_1 - t_2|/m) \qquad \tau_{corr} = m/k_S$$

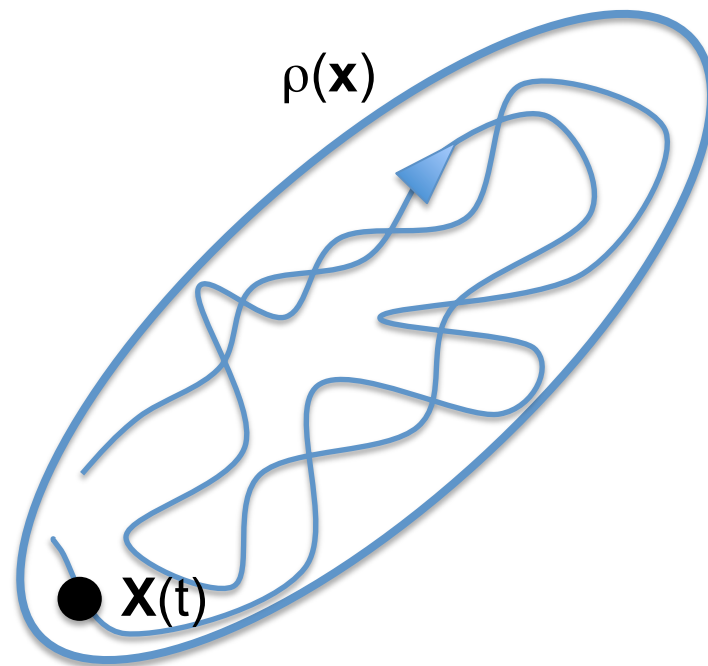$$D = \frac{RT}{6N_A\pi\mu a} \qquad \text{EINSTEIN RELATION}$$

Velocity response to applied force

Time scale of fluctuations $\qquad \tau_{corr} = m/k_S$ _____
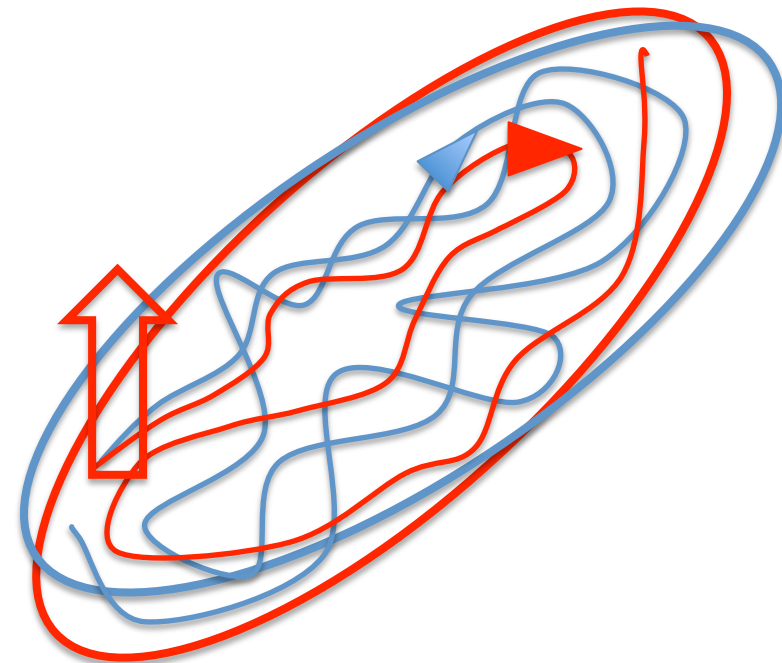
Applied force

FLUCTUATION -- DISSIPATION

"Dynamical systems" approach

Calculation of change in statistical measure of chaotic/random
system due to applied perturbation

$\rho(\mathbf{x})$

$\mathbf{X}(t)$

UNDISTURBED SYSTEM

DISTURBED SYSTEM

Evolution equation $\dfrac{d\mathbf{X}}{dt} = \mathbf{U}(\mathbf{X}, t)$

$\mathbf{U}(\mathbf{X}, t)$ is usually nonlinear and could contain explicit randomness

Equilibrium statistical properties described by probability density function $\rho(\mathbf{x})$

Perturb evolution equation $\dfrac{d\mathbf{X}}{dt} = \mathbf{U}(\mathbf{X}, t) + \Delta\mathbf{F}(\mathbf{X}, t)$

What is new $\rho(\mathbf{x})$?

[Perturbations to individual trajectories are large – perturbations to overall statistics are small.]

Consider small applied forcing $\Delta \mathbf{F} = \mathbf{f}(\mathbf{x})\delta(t)$

At $t=0$: $\rho(\mathbf{x}) \to \rho_+(\mathbf{x}) \simeq \rho(\mathbf{x}) - \nabla_{\mathbf{x}}.(\mathbf{f}(\mathbf{x})\rho(\mathbf{x}))$

$$\langle \phi(\mathbf{X}(\tau)) \rangle_{\mathbf{f}} = \int d\mathbf{x} \int d\mathbf{y} \phi(\mathbf{y}) \mathcal{P}(\mathbf{X}(\tau) = \mathbf{y}|\mathbf{X}(0) = \mathbf{x})\rho_+(\mathbf{x})$$

Compare identity
$$\langle \phi(\mathbf{X}(\tau))\psi(\mathbf{X}(0)) \rangle =$$
$$\int d\mathbf{x} \int d\mathbf{y} \phi(\mathbf{y}) \mathcal{P}(\mathbf{X}(\tau) = \mathbf{y}|\mathbf{X}(0) = \mathbf{x})\psi(\mathbf{x})\rho(\mathbf{x})$$

Hence $\quad \langle \phi(\mathbf{X}(\tau)) \rangle_{\mathbf{f}} \simeq \langle \phi(\mathbf{X}(\tau)) \frac{\rho_+(\mathbf{x})}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)} \rangle \quad$ and

$$\Delta\langle \phi(\mathbf{X}(\tau)) \rangle = -\langle \phi(\mathbf{X}(\tau)) \frac{\nabla_{\mathbf{x}}.(\mathbf{f}(\mathbf{x})\rho(\mathbf{x}))}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)} \rangle$$

Fluctuation-Dissipation Theorem

For steady **x**-independent $\Delta\mathbf{F}$

$$\Delta\langle\mathbf{X}\rangle = -\int_0^\infty d\tau \langle\mathbf{X}(\tau)\frac{\nabla_\mathbf{x}.\rho(\mathbf{x})}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)}\rangle.\Delta\mathbf{F} = \mathbf{L}\Delta\mathbf{F}$$

A:
General

If **X** is Gaussian then

$$\Delta\langle\mathbf{X}\rangle = \int_0^\infty d\tau \mathbf{C}(\tau)\mathbf{C}(0)^{-1}.\Delta\mathbf{F} = \mathbf{L}\Delta\mathbf{F}$$

B:
Gaussian

where $\mathbf{C}(\tau) = \langle\mathbf{X}(\tau)\mathbf{X}(0)\rangle$

Linear operator $\mathbf{L}$ given in terms of properties of undisturbed system

Crude approximation $\quad \Delta\langle\mathbf{X}\rangle \sim \tau_{\text{correlation}} \times \Delta\mathbf{F}$
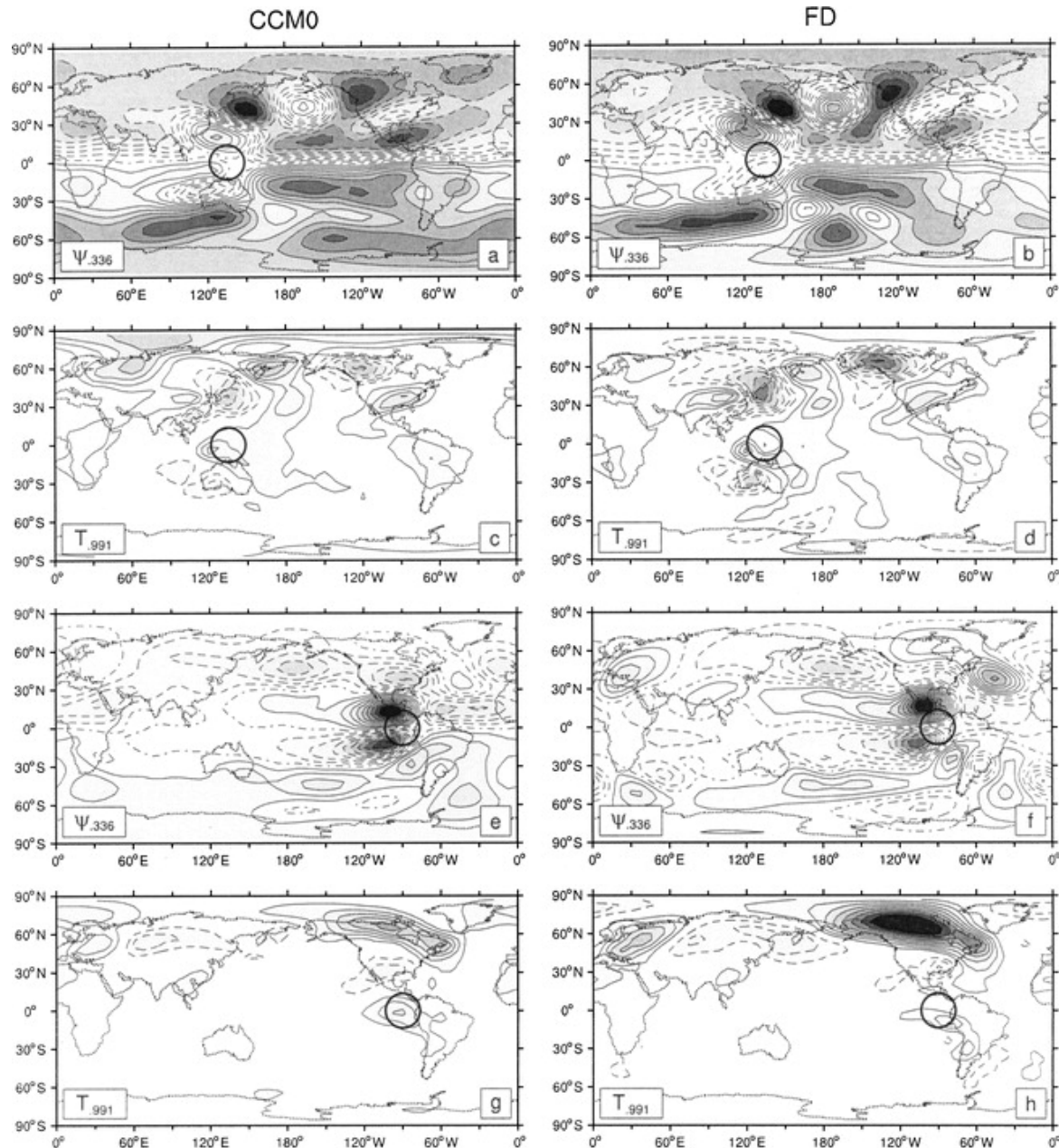
Gritsun and Branstator
(2007)

Application of Gaussian
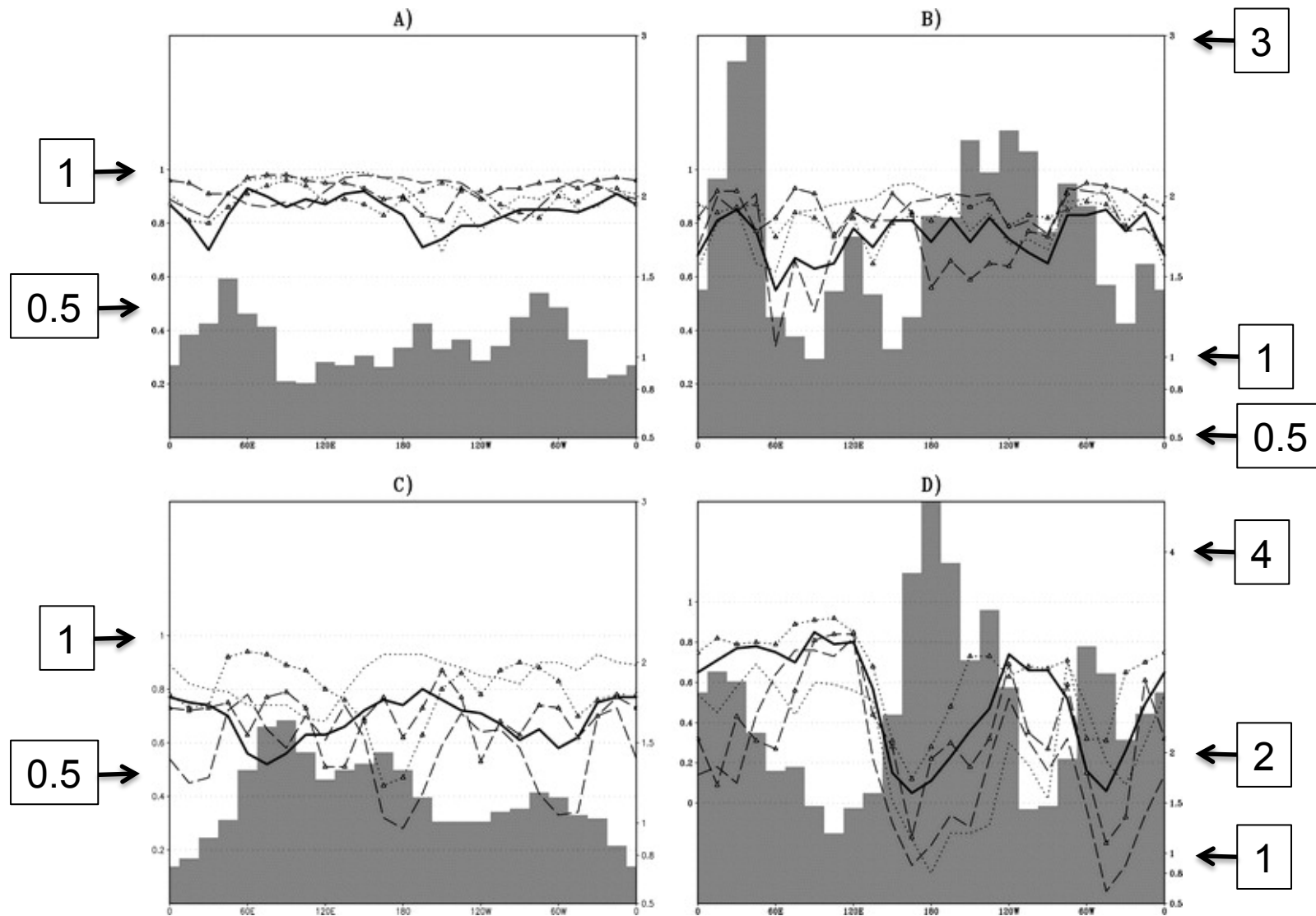FDT to predict response
to localised tropical
heating in GCM
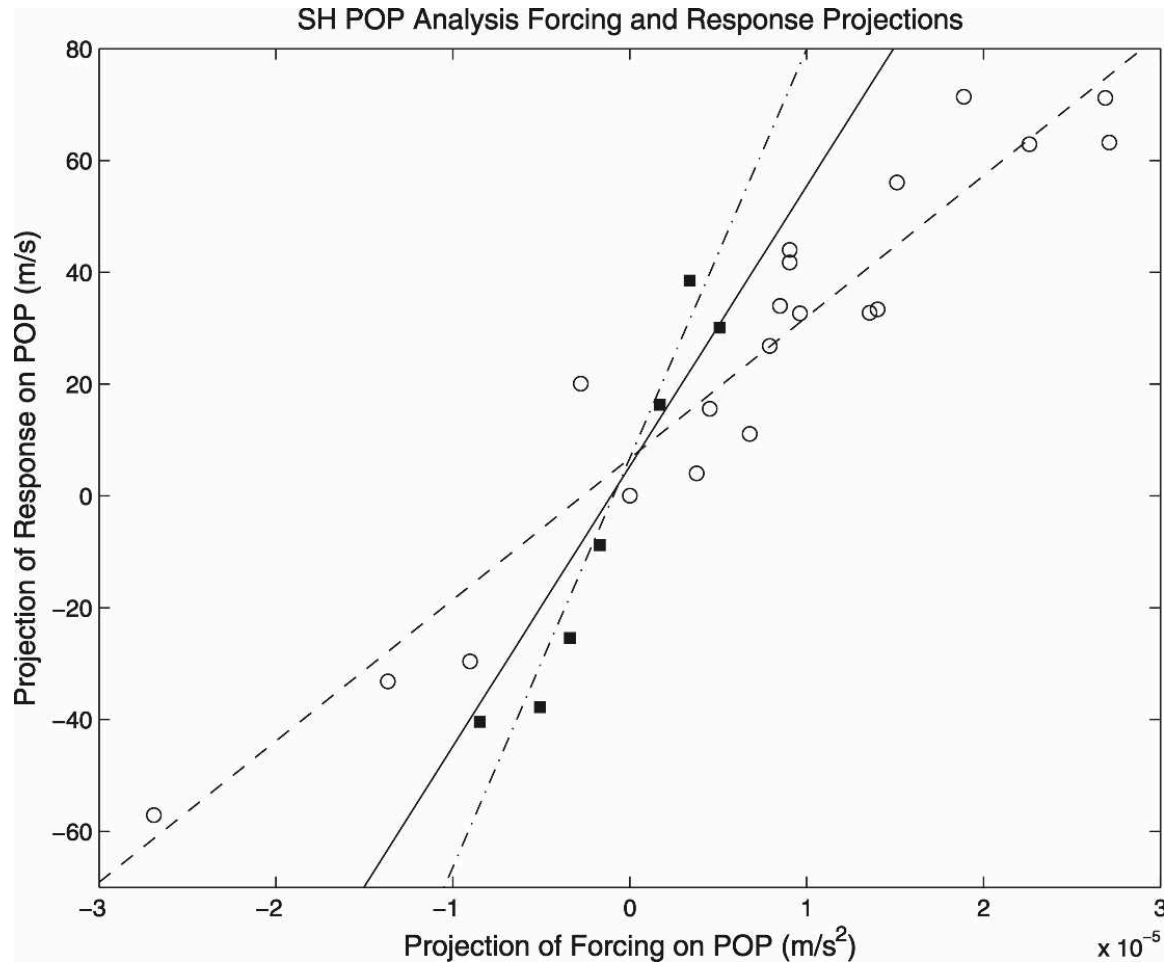
Individual AGCM
integrations 40000 days

FDT estimate
constructed from 4M day
integration

Gritsun and Branstator (2007)

Success of FDT measured by pattern correlation and amplitude ratio.

SH POP Analysis Forcing and Response Projections

Ring and Plumb (2008): Gaussian FDT makes incorrect prediction for response to zonally symmetric thermal (■) and mechanical (O) forcings

Practical issues in applying the FDT

$$\mathbf{L} = \int_0^\infty d\tau\, \mathbf{C}(\tau)\, \mathbf{C}(0)^{-1}$$

- EOFs (which diagonalise $\mathbf{C}(0)$) are a natural choice of variable (but not the only possible choice)
- $<\mathbf{C}(\tau)\,\mathbf{C}(0)^{-1}>$ must be estimated from available data.
- $\mathbf{C}(0)^{-1}$ potentially ill-conditioned – number of useful EOFs may be restricted by length of data series
- integration from $\tau = 0$ to $\tau = \infty$ must be approximated by finite sum

Statistical requirements on application of Gaussian FDT

Cooper and H 2012
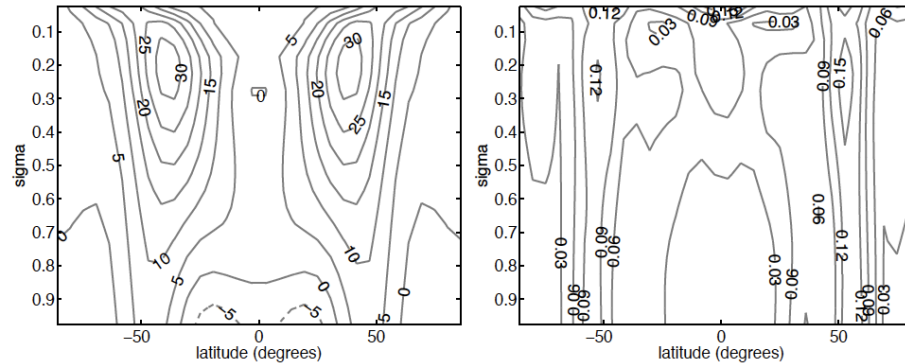
2-D linear model
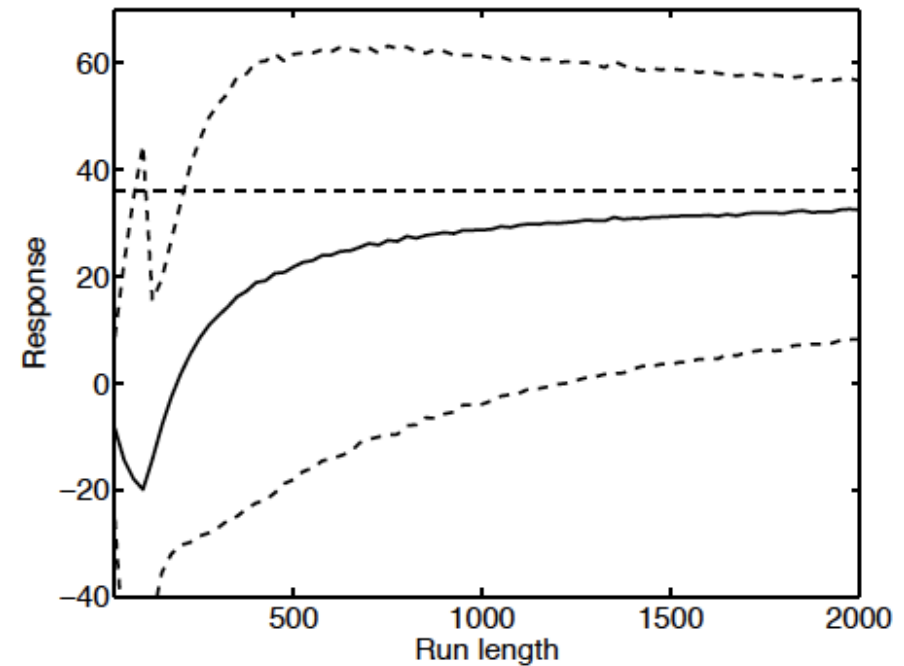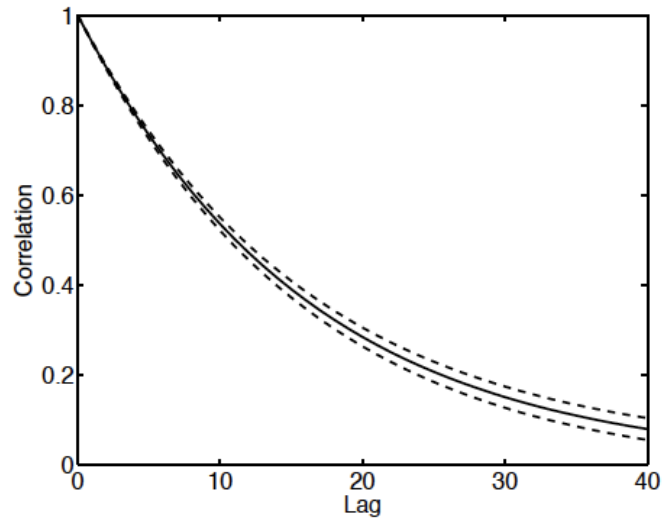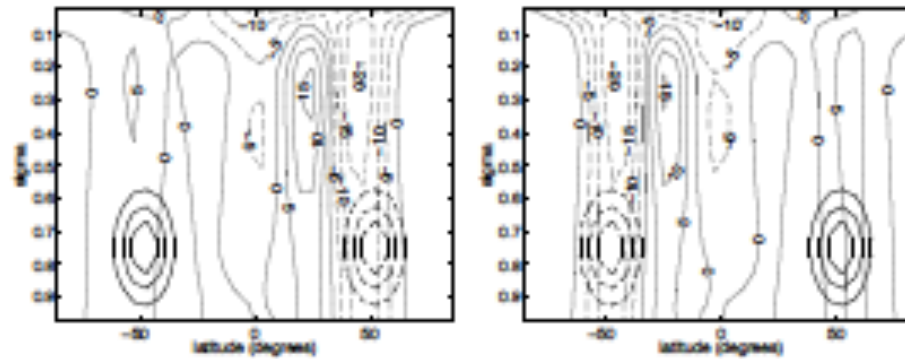

FDT prediction of response

Study based on simple T21L20 general circulation model (Cooper and H 2012)
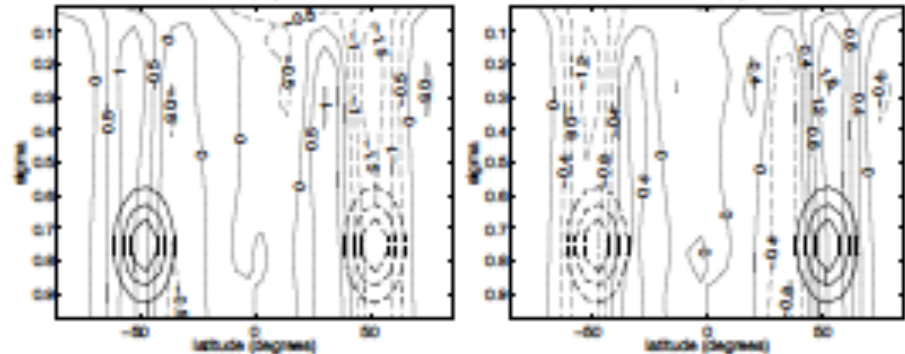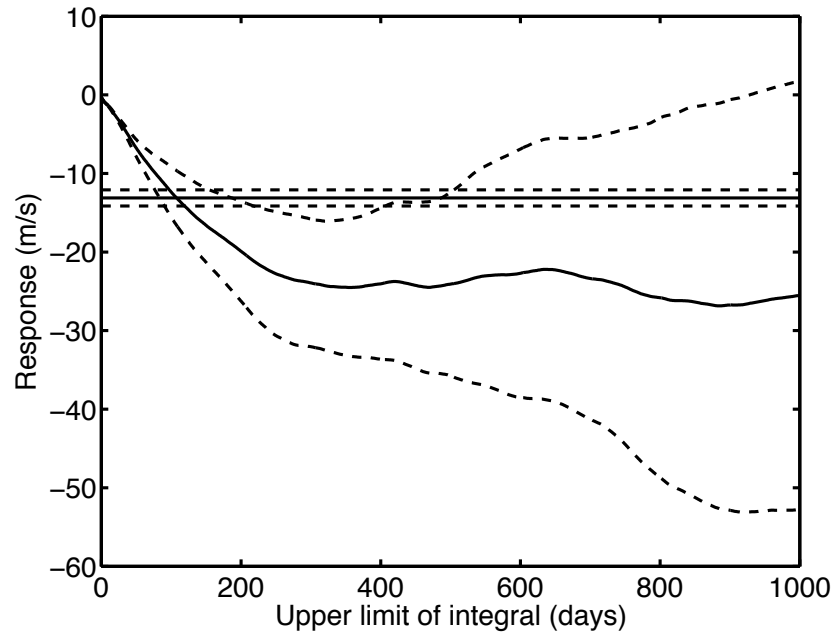

10000 day simulations, mean and variance
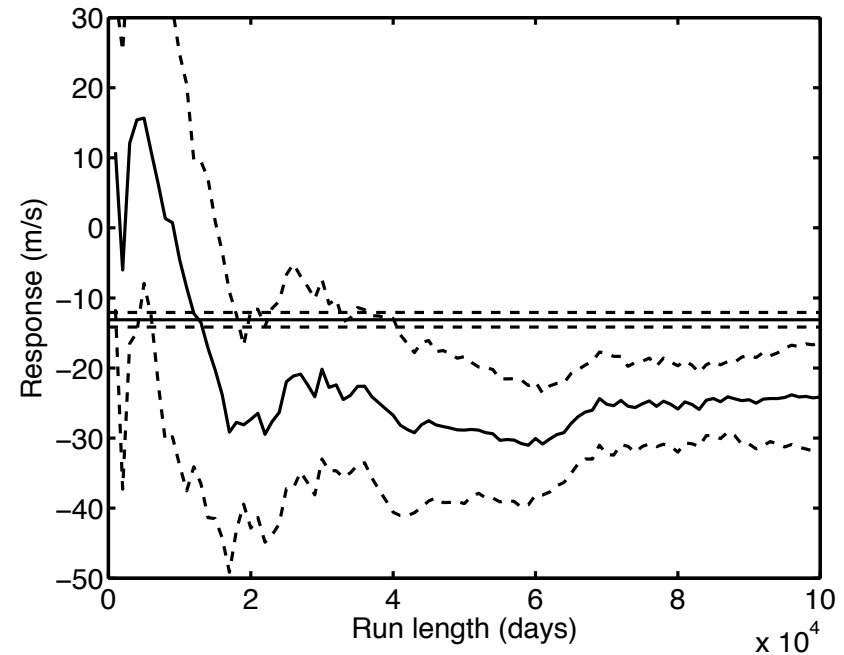
Forced response +/- 1 m/s/day

Forced response +/- 0.1 m/s/day

Application of the FDT to predict the response to forcing of a simple T21L20 general circulation model

(10 x 10^5 day integrations)

(300 day upper limit to integral)

What is the
optimal number
of EOFs to
include in the
calculation?

(T31L20 model)
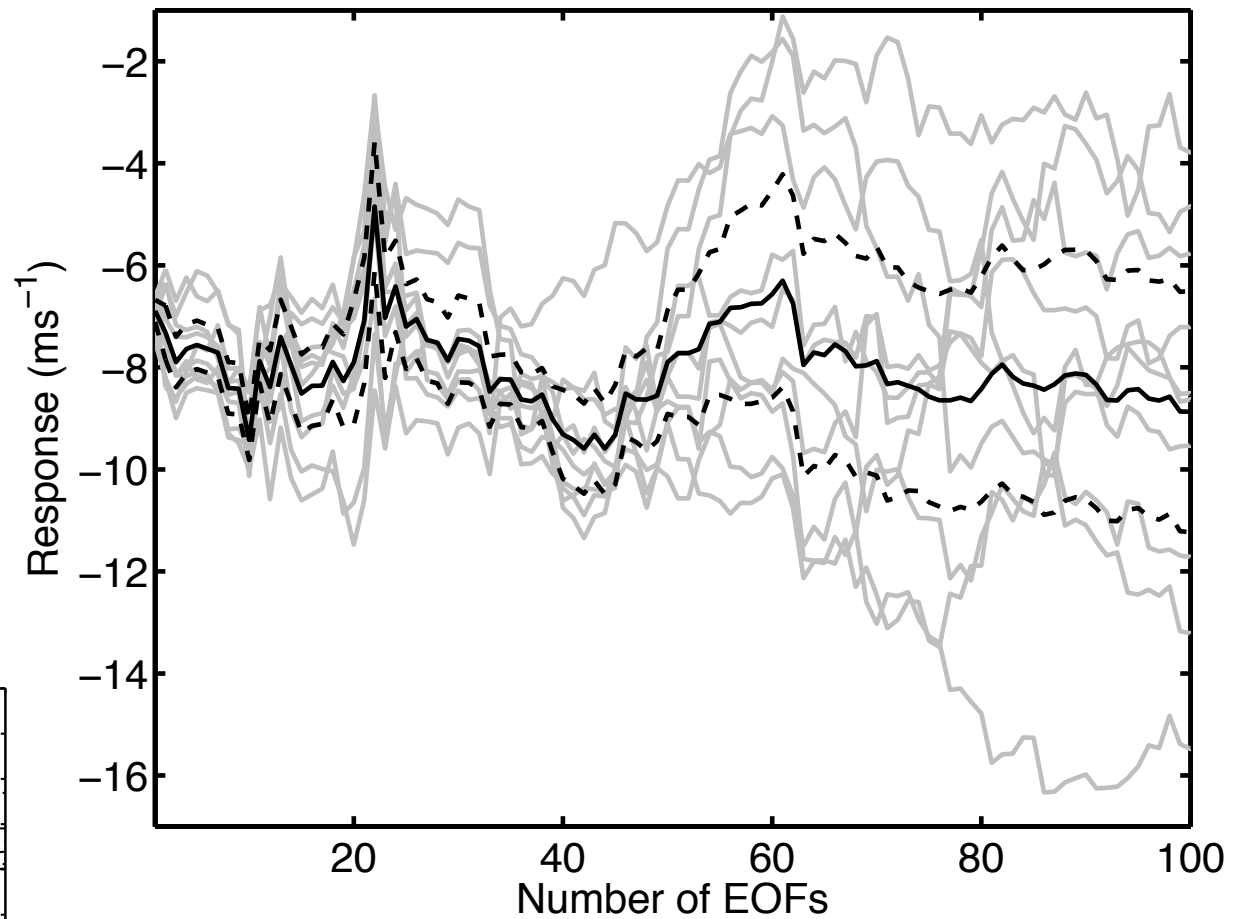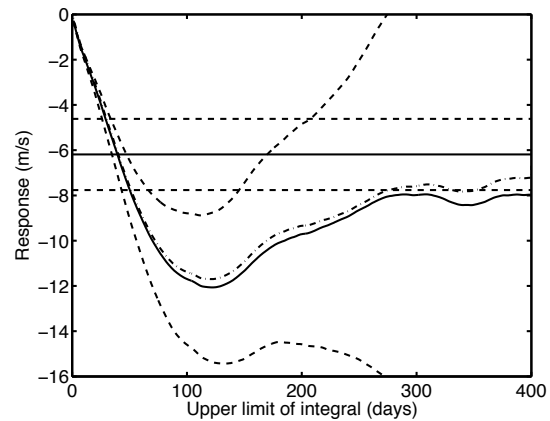
Specification of forcing

$$\mathbf{L}_{\text{Gaussian}} = \int_0^\infty d\tau \, \mathbf{C}(\tau) \mathbf{C}(0)^{-1}$$

Where did the equations go?

Implications of truncation

e.g. Ring and Plumb (2007)

$$\frac{\partial u}{\partial t} + \cdots = \mathcal{F} \qquad \frac{\partial T}{\partial t} + \cdots = 0$$

$$\frac{\partial u}{\partial t} + \cdots = \mathcal{L}(u, \mathcal{F})$$
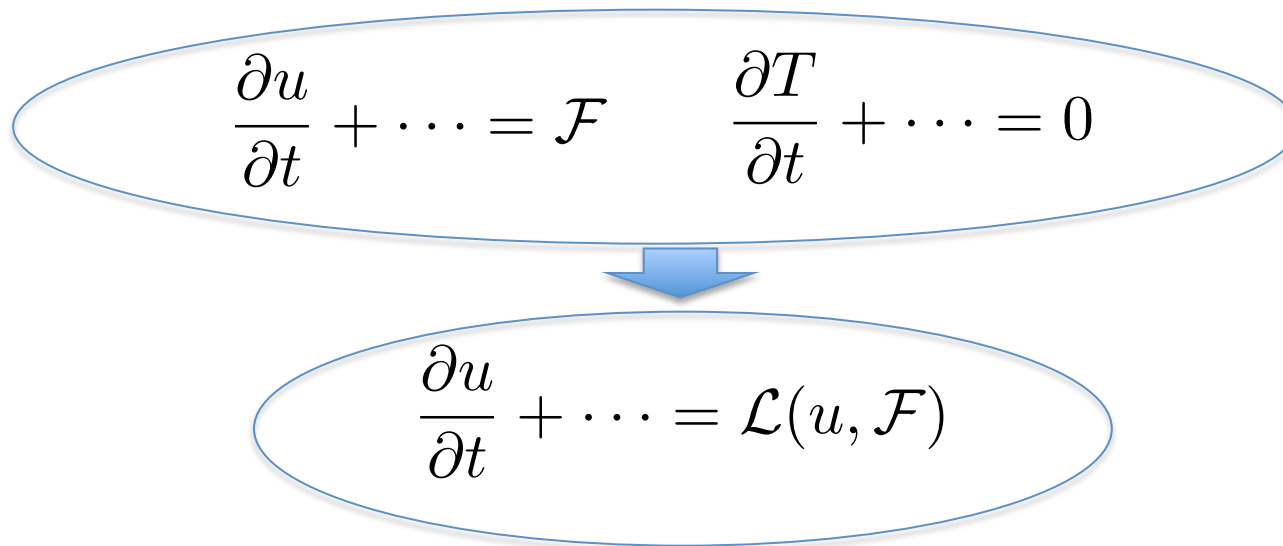
Effect of using
climatological u
versus u=0 in

$$\mathcal{L}(u, \mathcal{F})$$
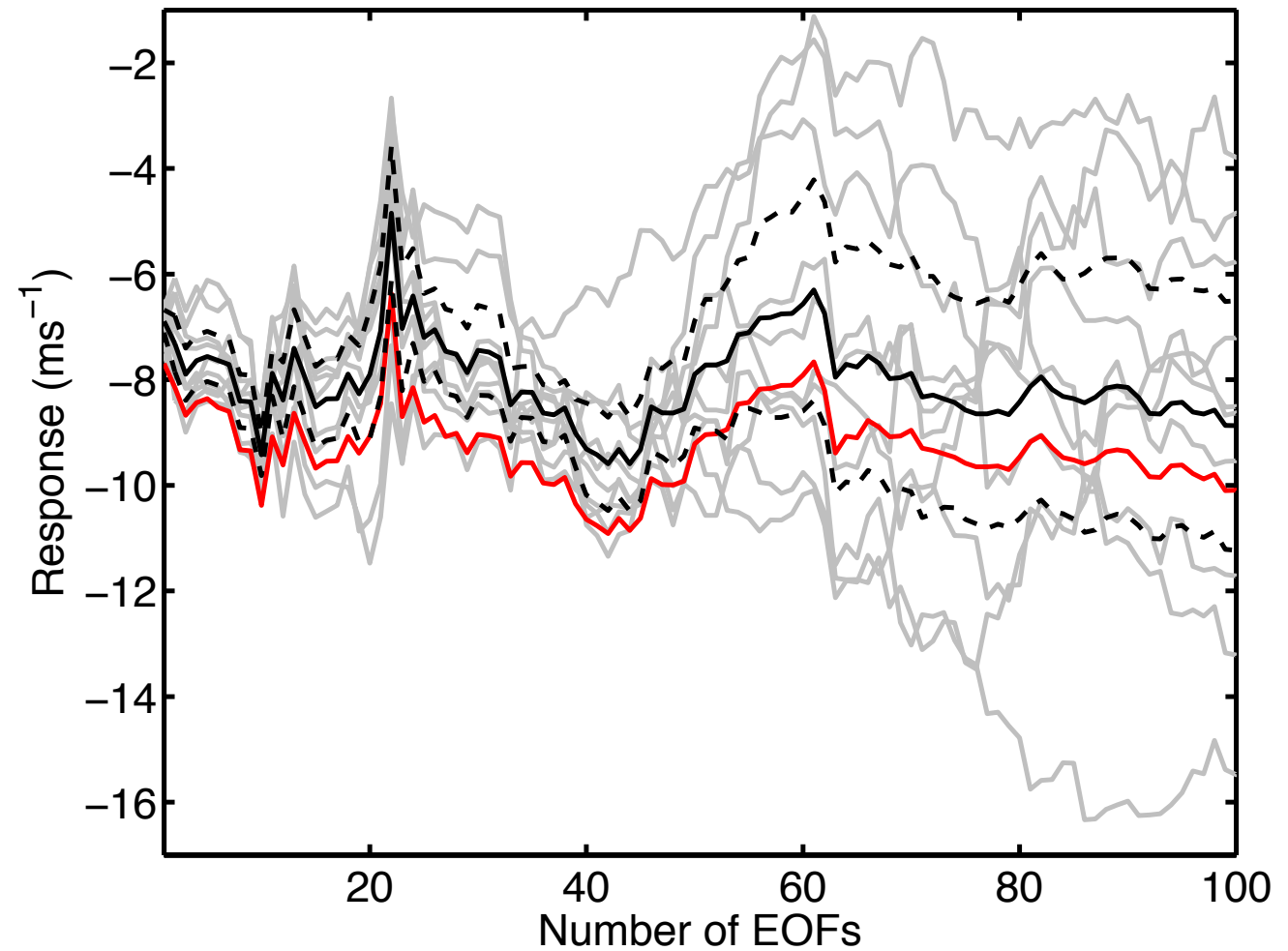
Usefulness of Gaussian
FDT may be limited by
non-Gaussianity?
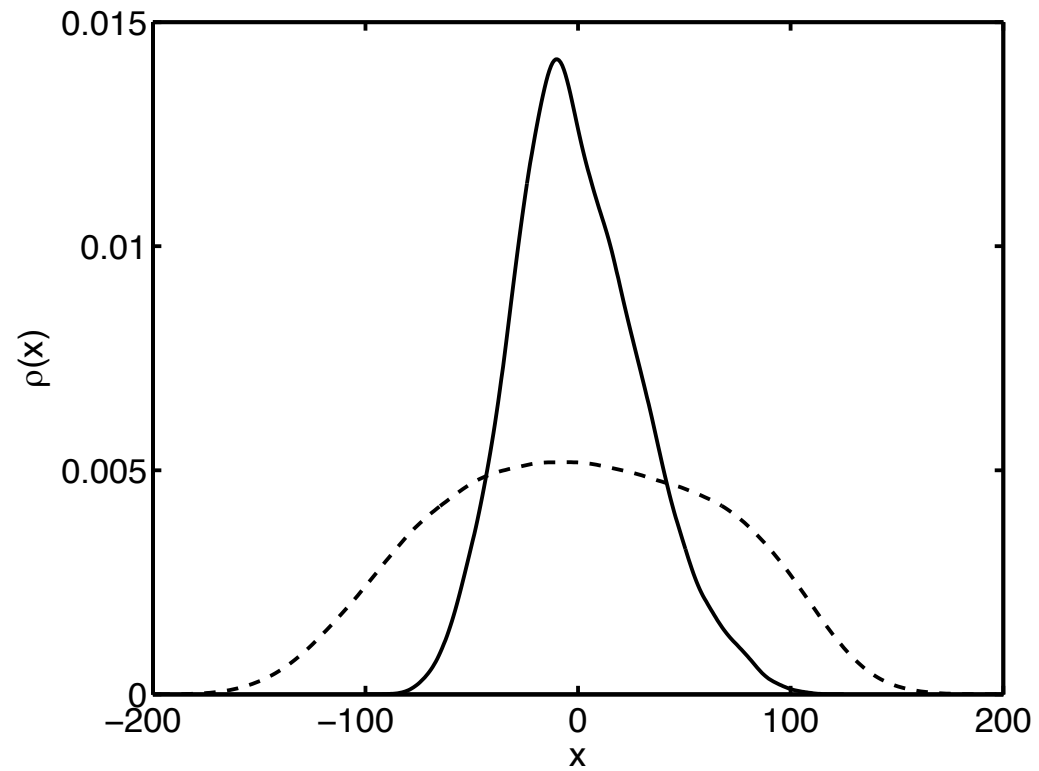
pdfs of EOF1 in zonal
wind for T21L20 and
T31L20 simulations

The non-Gaussian case – a 'non-parametric FDT'

Cooper and H 2011

$$\Delta\langle \mathbf{X} \rangle = -\int_0^\infty d\tau \langle \mathbf{X}(\tau) \frac{\nabla_{\mathbf{x}}\rho(\mathbf{x})}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)} \rangle.\Delta\mathbf{F}$$

Estimate using kernel density estimator
method of non-parametric statistics

$$\hat{\rho}(\mathbf{x};h,N) = \frac{1}{Nh^d} \sum_{i=1}^{N} K(\frac{\mathbf{x} - \mathbf{X}_i}{h})$$

$$\nabla_{\mathbf{x}}\hat{\rho}(\mathbf{x};h,N) = \ldots$$

Simplest choice for K(.) is
isotropic Gaussian

Bias and *Uncertainty* depend on *h* and *N*.

strongly non-Gaussian test case

$$\frac{dX}{dt} = b_1 X - b_2 X^3 + \xi$$

Perturbed
and
unperturbed
pdfs
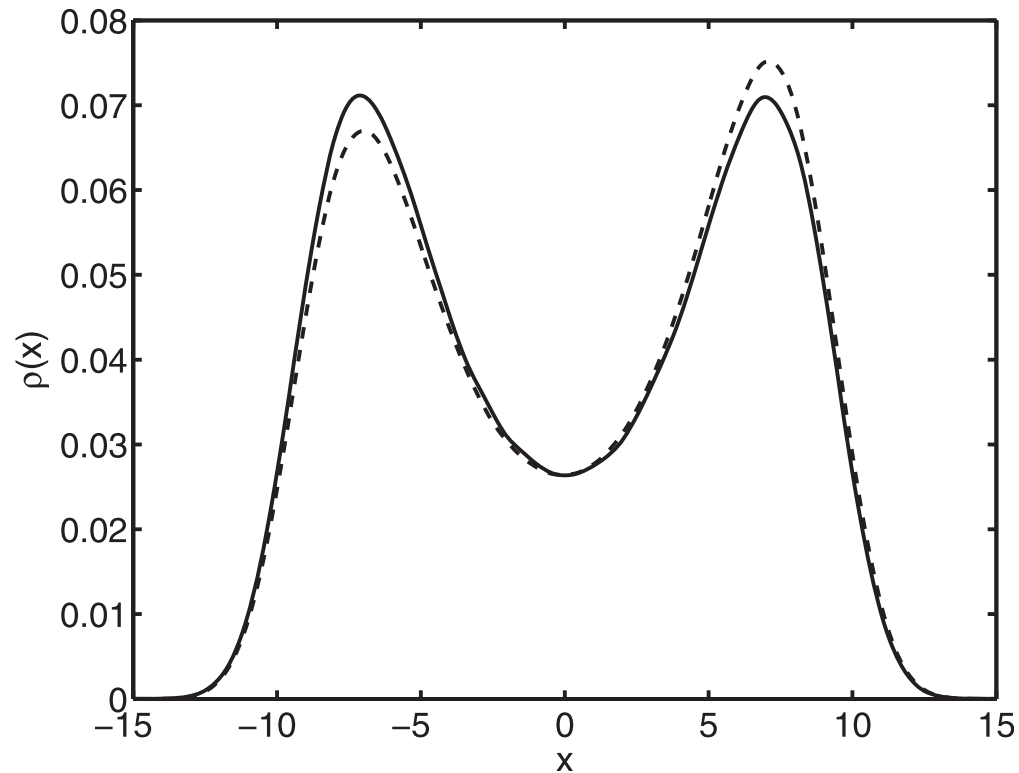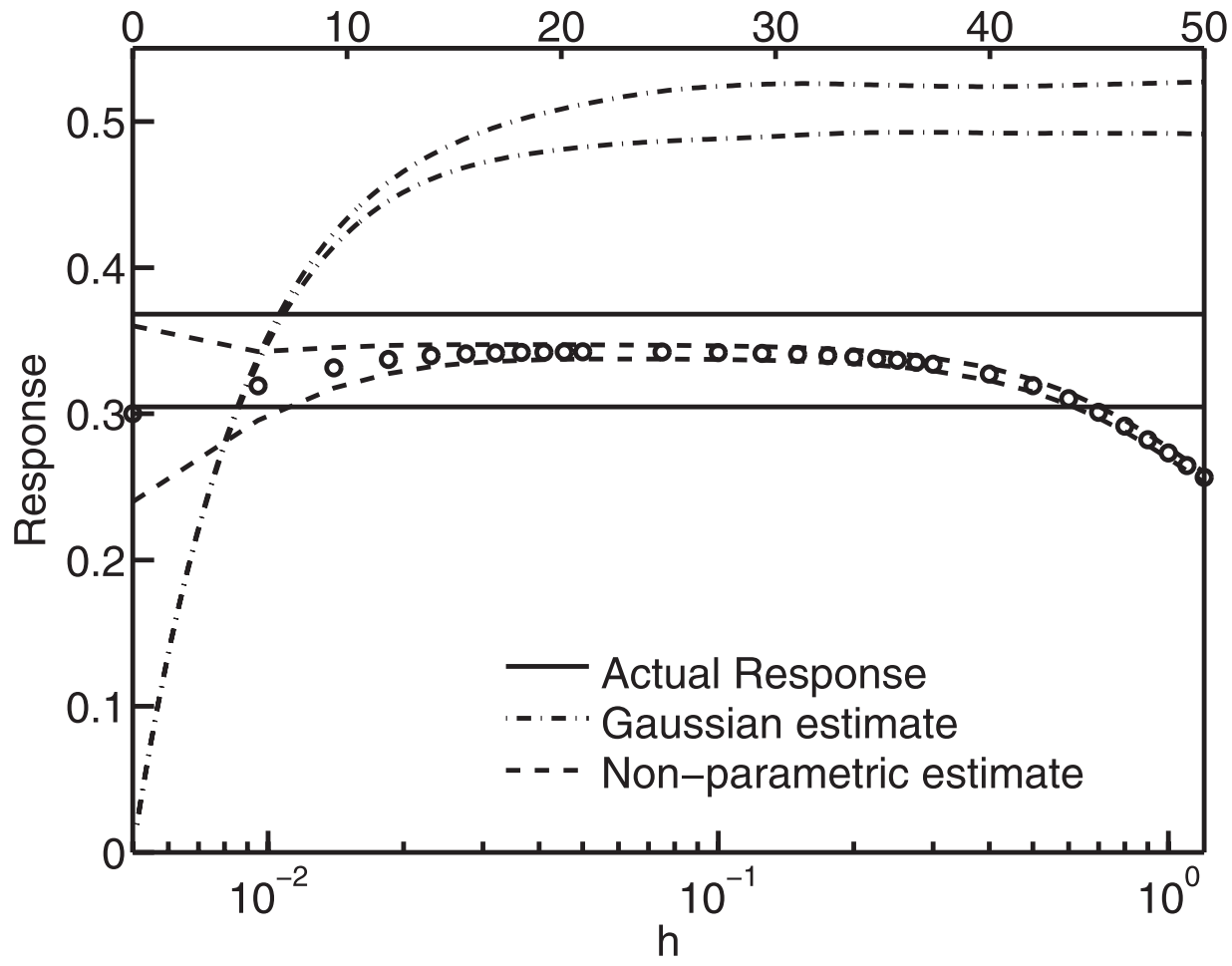
Application to stochastic Lorenz 1963 model
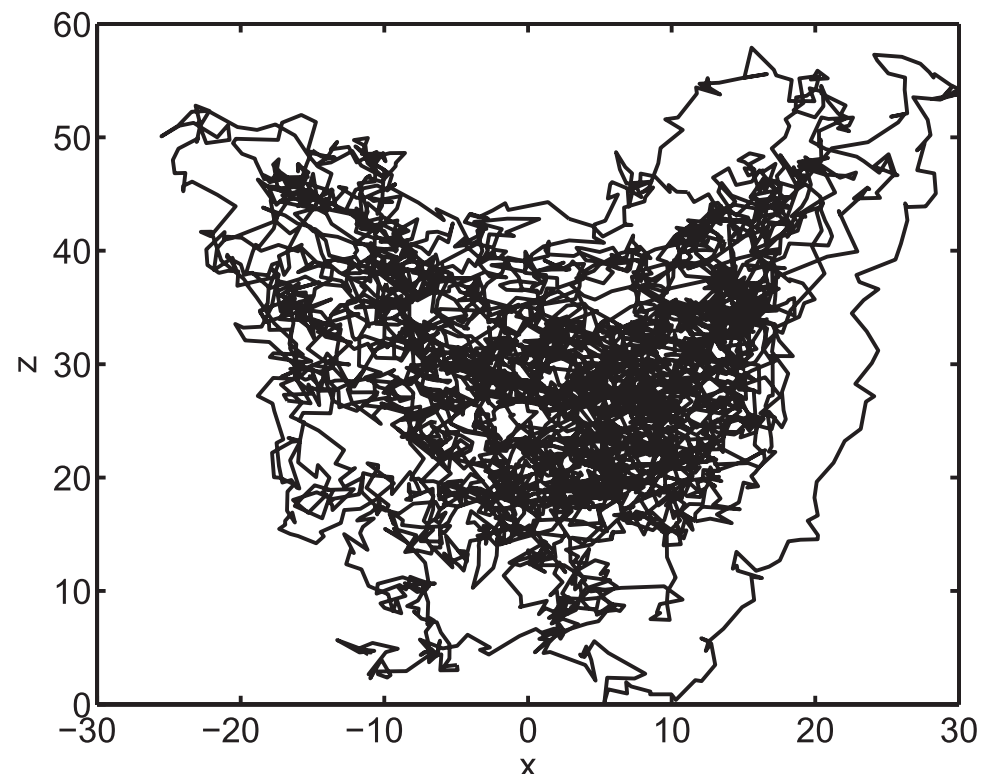
Compare with Thuburn (2005) approach of solving Fokker-Planck equation

Response in x direction to forcing applied in different directions
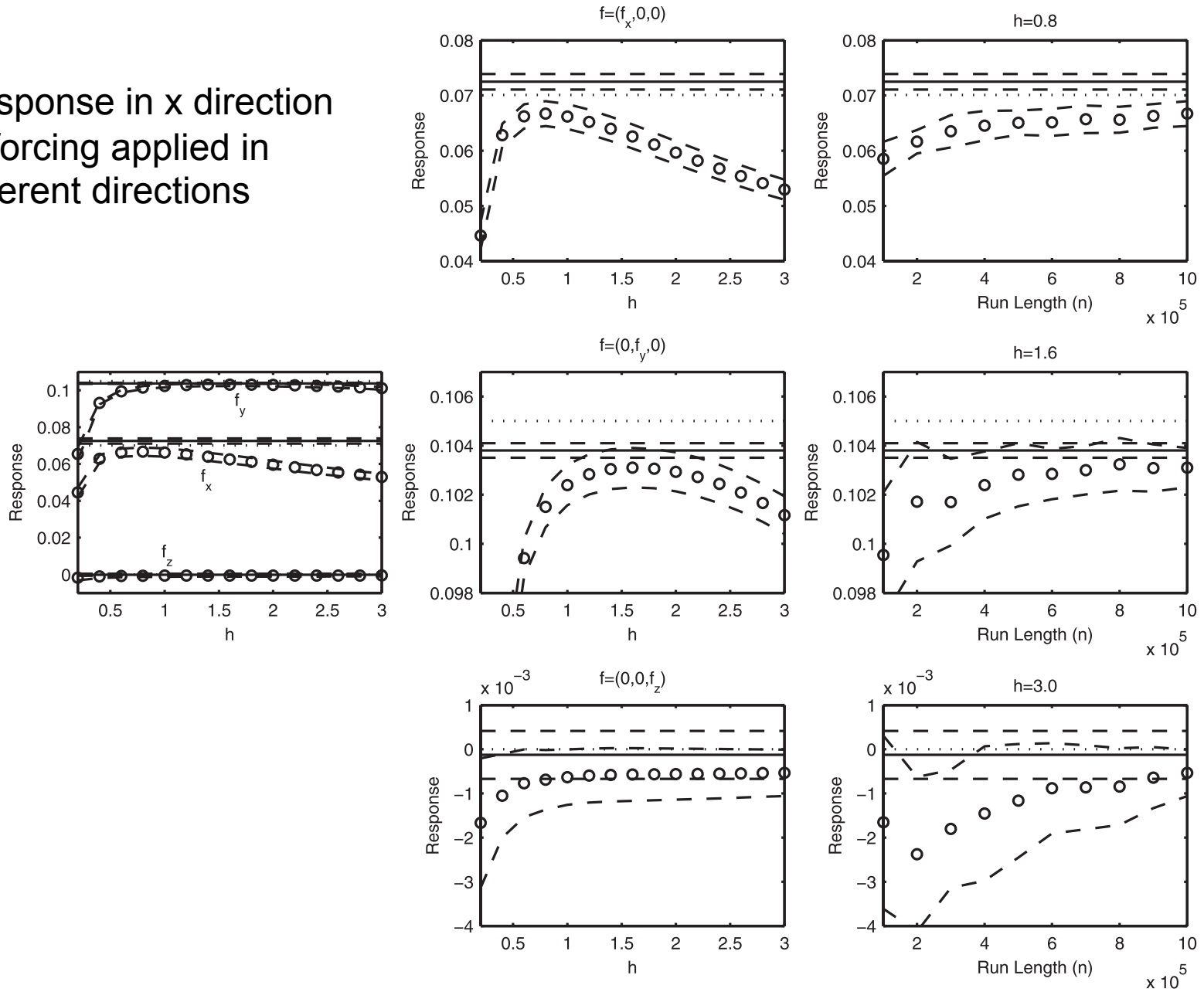
What is a useful measure of
non-Gaussianity?
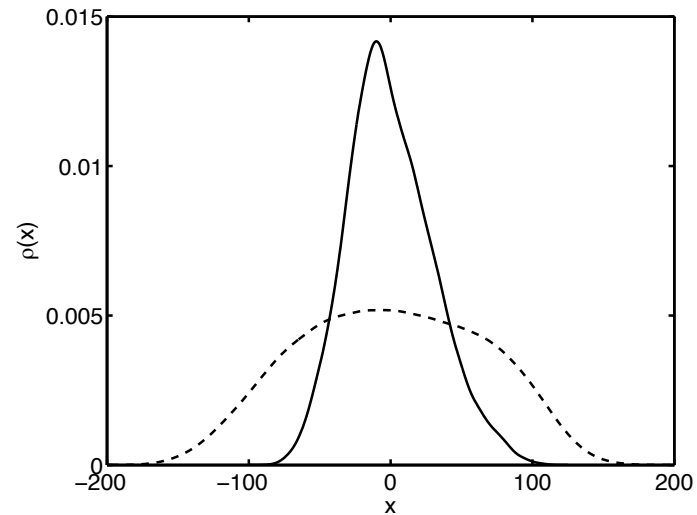
$$\mathbf{L} - \mathbf{L}_{Gaussian} = \int_0^\infty \langle\langle \mathbf{X}(\tau)|\mathbf{X}(0)\rangle \{-\rho(\mathbf{x})^{-1}\nabla_{\mathbf{x}}\rho(\mathbf{x})|_{\mathbf{x}=\mathbf{X}(0)} - \mathbf{X}(0).\mathbf{C}(0)^{-1}\}\rangle$$

Depends on structure of time
correlations as well as form of pdf

Summary

- FDT potentially provides a quantitative description of tropospheric response to forcing (e.g. ozone hole, solar cycle, greenhouse gas increase) given information on statistics of unforced circulation

- If model low-frequency variability (timescales and patterns) is wrong then response to forcing will be wrong

- Typical response to forcing will be leading singular vector of response operator (providing forcing has significant projection onto leading singular vector), not necessarily the leading EOF.

- In practice can FDT do better than simple estimation of timescale of leading EOF?

- Applications? Model assessment/intercomparison?

FDT confusion?

FDT implies that response to forcing will have the same form as leading EOF? Not necessarily!

$$d\mathbf{X} = \mathbf{B}\mathbf{X}\,dt + d\xi \qquad\qquad \langle d\xi\,d\xi \rangle = \mathbf{Q}\,dt$$

$$\mathbf{B}\mathbf{C}(0) + \mathbf{C}(0)\mathbf{B}^T + \mathbf{Q} = 0 \qquad \boxed{\mathbf{C}(0) \text{ depends on } \mathbf{Q}}$$

Analogy to system of springs – with weakest spring responding preferentially?

Better to consider mixing in dynamical system – the longest-lived structure is the one that mixes least rapidly.

Deser et al (2004) – heuristic approach to response problem

(a)

	Total Response	Internal Mode Projection	Residual
300mb			
700mb			
1000mb			

'INDIRECT'   'DIRECT'
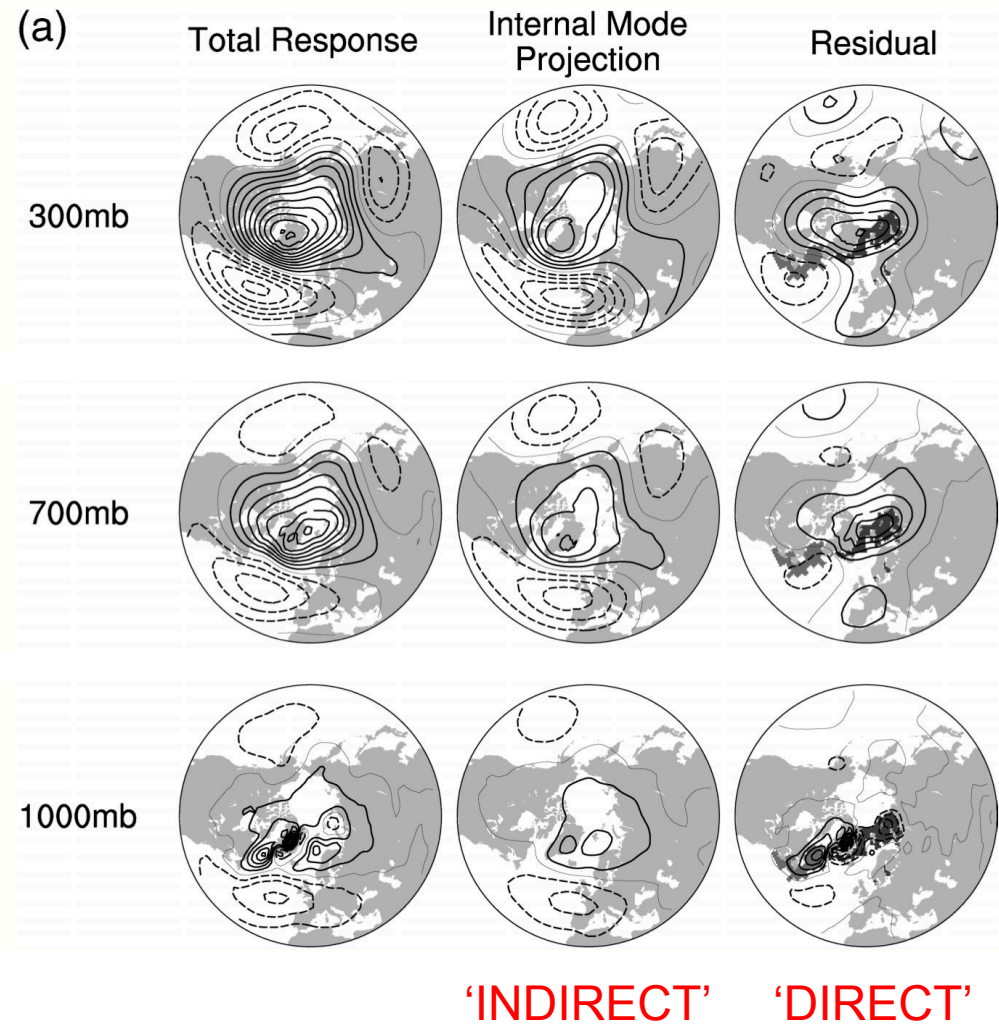
Future lines of work? (Is the FDT a practical quantitative tool?)

- Statistical nature of FDT requires explicit information on/ estimates for bias and uncertainty

- Non-gaussian extension of FDT potentially extends validity (but there are challenges in implementation – can we escape the 'curse of dimensionality' or avoid it by working in a truncated system?)

- FDT for truncated system is non-trivial – need to consider proper 'effective forcing' on truncated system.

- Clearer practical guide to implementation of FDT (How long a data record is needed for required precision? How many degrees of freedom to include?)

Usefulness of linear theory?

depends on problem being considered – but recall Haigh et al (2005)
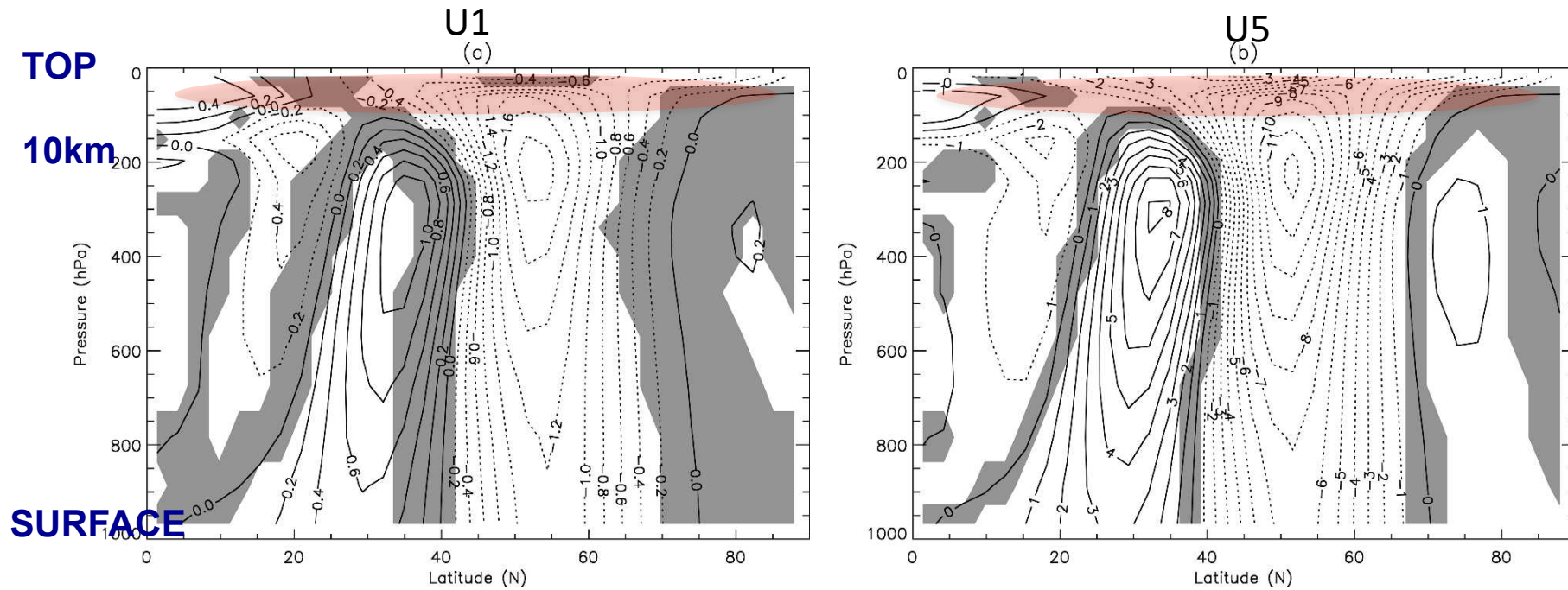
Kidston et al (2015): tropospheric response to stratosphere on different timescales
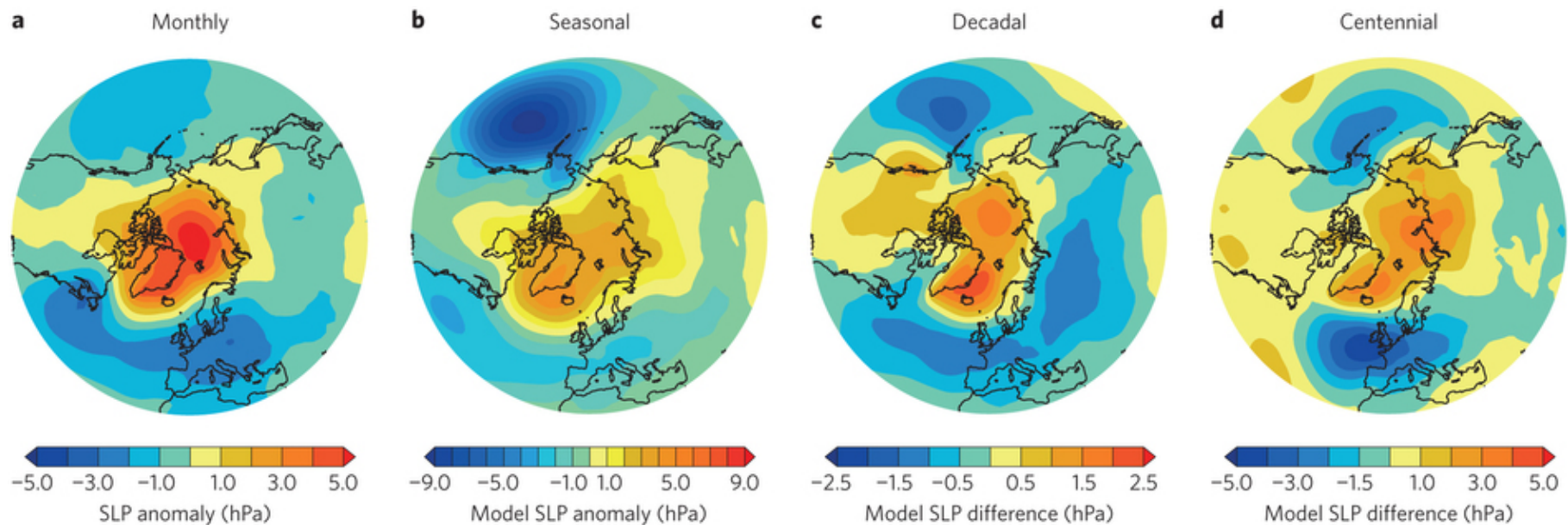
FDT – cultural differences

20th century physics: large systems, small fluctuations, FDT has been discussed/applied/interpreted in terms of macroscopic variables.

21st century physics: extension to non-equilibrium small systems with large fluctuations?

Dynamical systems: Formal derivation/justification of 'fluctuation-response' operators, conditions for applicability, can problems of non-smoothness/non-differentiability be overcome?

Climate/circulation: Evaluation of 'fluctuation-response' operator from model simulation (or from data?) is a problem in statistics of large-degree-of-freedom systems. How much data is needed for required accuracy? How can effective dimensionality be most effectively reduced?

THE END

Reference list for PHH Les Houches seminar

The following are papers referenced in the slides. I've added papers by Fuchs et al (2015) and Hassanzadeh and Kuang (2016a,b) since they add significantly to some of the things that I talked about.

Butler, A.H., Thompson, D. W. J., Heikes, R., 2010: The Steady-State Atmospheric Circulation Response to Climate Changelike Thermal Forcings in a Simple General Circulation Model. J. Climate, 23, 3474–3496, doi: 10.1175/2010JCLI3228.1.

Chen, G., Held, I.M., Robinson, W. A., 2007: Sensitivity of the Latitude of the Surface Westerlies to Surface Friction. J. Atmos. Sci., 64, 2899–2915, doi: 10.1175/JAS3995.1.

Cooper, F. C., Haynes, P. H., 2011: Climate sensitivity via a non-parametric fluctuation-dissipation theorem. J. Atmos. Sci. 68, 937-953.

Cooper, F. C., Haynes, P. H., 2012: Assessment of the fluctuation-dissipation theorem as an estimator of the response of the tropospheric response to forcing. (preprint). [This paper was never published. I'd be pleased to email a copy to anyone interested. Some of it has been significantly superseded by the papers by Fuchs et al (2015) and Hassanzadeh and Kuang (2016a,b).]

Deser, C., G. Magnusdottir, R. Saravanan, and A. Phillips, 2004: The effects of North Atlantic SST and sea ice anomalies on the winter circulation in CCM3. Part II: direct and indirect components of the response, J. Climate, 17, 877?889, doi:10.1175/1520- 0442(2004)017¡0877:TEONAS¿2.0.CO;2.

Fuchs, D., S. Sherwood, and D. Hernandez, 2015: An exploration of multivariate fluctuation dissipation operators and their response to sea surface temperature perturbations. J. Atmos. Sci., 72, 472–486, doi:10.1175/JAS-D-14-0077.1.

Gritsun, A., Branstator, G., 2007: Climate response using a three-dimensional operator based on the fluctuation-dissipation theorem. J. Atmos. Sci., 64, 2558–2575.

Haigh, J. D., Blackburn, M., and Day, R. 2005: The response of tropospheric circulation to perturbations in lower-stratospheric temperature. J. Climate, 18, 3672–3685, doi:10.1175/ JCLI3472.1.

Hassanzadeh P. and Kuang Z., 2016: The linear response function of an idealized atmosphere. Part I: Construction using Green's functions and applications, Journal of the Atmospheric Sciences, 79, 3423-3439.

Hassanzadeh P., Kuang Z., 2016: The linear response function of an idealized atmosphere: Part II: Implications for the practical use of the Fluctuation-Dissipation Theorem and the role of operator's nonnormality. Journal of the Atmospheric Sciences, 79, 3441-3452.

Kidston, J., Scaife, A. A., Hardiman, S. C., Mitchell, D. M., Butchart, N., Baldwin, M. P., Gray, L. G., 2015: Stratospheric influence on tropospheric jet streams, storm tracks and surface weather. Nature Geoscience, 8, 433-440.

Kushner, P. J., and Polvani, L. M. , 2004: Stratospheretroposphere coupling in a relatively simple AGCM: The role of eddies. J. Climate, 17, 629–639, doi:10.1175/1520-0442(2004)017,0629: SCIARS.2.0.CO;2.

Lu, J., Chen, G., Frierson, D.M.W, 2008: Response of the Zonal Mean Atmospheric Circulation to El Niño versus Global Warming. J. Climate, 21, 5835–5851, doi: 10.1175/2008JCLI2200.1.

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